JOURNAL OF COMBINATORIAL THEORY, Series A 72, 209–231 (1995)

A Generalization of Semimodular Supersolvable Lattices

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Communicated by the Managing Editors

Received July 29, 1993

Stanley (*Algebra Universalis* **2**, 1972, 197–217) introduced the notion of a supersolvable lattice, L, in part to combinatorially explain the factorization of its characteristic polynomial over the integers when L is also semimodular. He did this by showing that the roots of the polynomial count certain sets of atoms of the lattice. In the present work we define an object called an atom decision tree. The class of semimodular lattices with atom decision trees strictly contains the class of supersolvable lattices, but their characteristic polynomials still factor for combinatorial reasons. We then apply this notion to prove the factorization of polynomials associated with various hyperplane arrangements having non-supersolvable lattices.

1. ATOM DECISION TREES

In this section we will introduce our main object of study: atom decision trees. We will show that the characteristic polynomial for a semimodular lattice admitting an atom decision tree has non-negative integral roots. In fact, these roots count the sizes of certain sets of atoms of the lattice. We will also note how the semimodular supersolvable lattices of Stanley [18] have atom decision trees and so are a special case. First, however, we must give some definitions and notation. Any terms not defined can be found described in Stanley's book [19].

Let L be a lattice with meet and join denoted by $\wedge$ and $\vee$, respectively. All our lattices will be finite having a minimal element $\hat{0}$ and maximal

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element $\hat{1}$. The *Möbius function* of L is defined inductively on elements $x \in L$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0} \\ -\sum_{y<x} \mu(y) & \text{if } x > \hat{0}. \end{cases}$$

The Möbius function is one of the fundamental invariants of L. Now suppose L is graded with the rank of $x \in L$ denoted by $\mathrm{rk}(x)$. Then the *characteristic polynomial of* L is

$$\chi(L, t) = \sum_{x \in L} \mu(x)\, t^{\mathrm{rk}(\hat{1}) - \mathrm{rk}(x)}. \tag{1}$$

One uses the corank of x, rather than its rank, as the exponent on t so that the polynomial will be monic. Since the characteristic polynomial is just the generating function for the Möbius function, it is also of fundamental importance.

Now suppose L is semimodular so that

$$\mathrm{rk}(x) + \mathrm{rk}(y) \geqslant \mathrm{rk}(x \wedge y) + \mathrm{rk}(x \vee y) \qquad \text{for all} \quad x, y \in L, \tag{2}$$

and consider the set A of atoms of L. Given a subset $B \subseteq A$, we define

$$x_B = \bigvee_{b \in B} b. \tag{3}$$

Then B is *independent* if

$$\mathrm{rk}(x_B) = |B|,$$

where $|\cdot|$ denotes cardinality; otherwise B is *dependent.* If B is independent then we say that B is a *base* for x_B. If B is a minimal (with respect to inclusion) dependent set then we say that B is a *circuit.* Given a total order on A, then a *broken circuit* is obtained by taking a circuit and removing its smallest atom. Rota [16] first stated an important theorem giving an interpretation to the Möbius function in terms of broken circuits. He did this for geometric lattices, i.e., those which are semimodular and where every element is a join of atoms. It is not hard to generalize this result to lattices which are just semimodular, the case that we will need. In fact this theorem can be generalized even further, as has been done by Sagan [17]. Related results can be found in the papers of Björner [2] and Björner and Ziegler [4].

THEOREM 1.1. *Let L be a finite semimodular lattice and suppose the atoms A of L are arranged in some total order. Then, for any $x \in L$,*

$$\mu(x) = (-1)^{\mathrm{rk}(x)} \text{(number of bases of } x \text{ that contain no broken circuit).} \tag{4}$$

The bases of this theorem are called *NBC* (*non-broken circuit*) *bases*. Furthermore, an NBC set (one containing no broken circuit) is automatically independent and so is a base. If L has additional structure, then there is a nice way of constructing such bases. A semimodular lattice L is *supersolvable* if it contains a maximal chain C, called an *M-chain*, such that every element of M is modular. Suppose L is supersolvable with M-chain $\hat{0} = x_0 < x_1 < \cdots < x_n = \hat{1}$ and let

$$A_i = \{a \in A : a \leqslant x_i \text{ but } a \nleqslant x_{i-1}\}.$$

Stanley [18] and independently Garsia and Wachs (unpublished) discovered the following theorem.

THEOREM 1.2. *Let L be a finite semimodular supersolvable lattice and consider any ordering of its atoms such that the elements of A_i come before those of A_{i+1} for all i. Then the corresponding NBC bases for L consist of all sets obtained by choosing at most one atom from each A_i and*

$$\chi(L, t) = \prod_i (t - |A_i|).$$

Thus the roots of $\chi(L, t)$ are positive integers since they are just the cardinalities of certain sets of atoms.

We can now describe the concept of an atom decision tree which is fundamental to all that follows. For any terms from graph theory that we do not define, see the text of Chartrand and Lesniak [7]. All our trees will be rooted, ordered, and edge-labeled. Each label will be an assignment statement (as used in computer science) which returns a set B of atoms, e.g., $B := B \cup \{a\}$, where a is an atom. Also, an edge from a vertex v to a child w of v will be said to *leave* v and *enter* w. Let $\lambda = (\lambda_1, \lambda_2, ...)$ be an ordered partition of positive integers. If L is a semimodular lattice then we say that L *admits an atom decision tree* (*ADT*), T, *of type* λ when the following two conditions hold.

1. Every vertex at level i of T has λ_i edges leaving it labeled with assignment statements that add exactly one atom to B and one edge that leaves B invariant ($B := B$).

2. For some atom ordering of L, every NBC base is obtained exactly once by starting with $B = \varnothing$ at the root and then performing the assignments on the edges of a path from the root of T to a leaf.

Note that by condition 1, all the leaves of T are at the same level.

As an example, consider the lattice, L, shown in Fig. 1. Order the atoms from left to right: $a < b < c < d$. Then L has only one circuit $\{a, b, c\}$ with

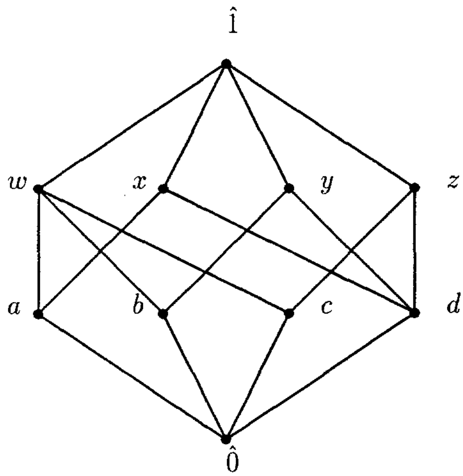

FIG. 1. A lattice L.

corresponding broken circuit $\{b, c\}$. The NBC bases of each element are given in the following table:

element	$\hat{0}$	a	b	c	b	w	x	y	z	$\hat{1}$
NBC bases	$\varnothing$	a	b	c	d	a, b a, c	a, d	b, d	c, d	a, b, d a, c, d

An ADT for L is given in Fig. 2. For simplicity, each assignment of the form $B := B \cup \{a\}$ (respectively, $B := B$) has been replaced by the edge-label a (respectively, $\varnothing$). In what follows, we will talk about the assignment statement label and its shorthand interchangeably.

For semimodular supersolvable lattices, there is a fixed partition of the atoms into subsets and the NBC bases are constructed by picking at most one atom from each subset. In the ADT case, the subsets correspond to vertices and the atoms chosen correspond to edges. The tree gives one

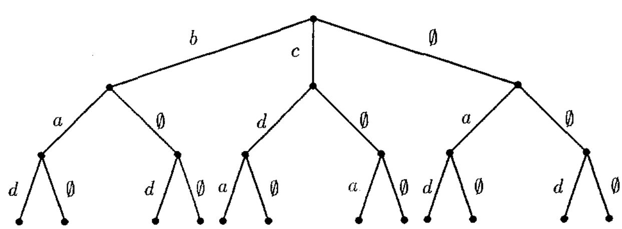

FIG. 2. An ADT, T.

more flexibility since the subset that you pick from next is allowed to vary depending on which atom, if any, you choose from the current subset. This situation resembles the relationship between edge-lexicographic and chain-lexicographic shellability [3].

The next proposition more exactly describes the connection between supersolvability and atom decision trees.

PROPOSITION 1.3. *Let L be a finite semimodular lattice. Then L supersolvable implies that L admits an ADT.*

Proof. Let the atoms of L be partitioned into subsets $A_1, A_2, \ldots$ as in Theorem 1.2. Construct a tree T satisfying condition 1 of the definition of an ADT so that the edges leaving a vertex at level i are labeled by the atoms of the set A_{i+1} and $\varnothing$. (The root is at level 0.) Then condition 2 follows directly from Theorem 1.2. ∎

Thus the semimodular supersolvable lattices are a subset of those admitting atom decision trees. We will see in the following sections that this containment is strict. We next show that the characteristic polynomial of a lattice with an ADT factors over the integers.

THEOREM 1.4. *Suppose L is a finite semimodular lattice admitting an ADT of type $\lambda = (\lambda_1, \lambda_2, \ldots)$. Then*

$$\chi(L, t) = \prod_i (t - \lambda_i).$$

Proof. By Theorem 1.1, the coefficient of t^{n-k} in $\chi(L, t)$ is just

$$(-1)^k \text{ (number of NBC bases of elements at rank } k \text{ in } L). \qquad (5)$$

Since L admits an ADT, T, each such base B can be constructed by choosing a path from the root of T to a leaf which chooses an edge labeled with an assignment augmenting B exactly k times. Because T has type λ this can be done in exactly $e_k(\lambda_1, \lambda_2, \ldots)$ ways, where e_k is the kth elementary symmetric function [13]. But then (5) coincides with the coefficient of t^{n-k} in $\prod_i (t - \lambda_i)$, so we are done. ∎

2. LATTICES FOR HYPERPLANE ARRANGEMENTS

We will now consider the semimodular lattices arising from hyperplane arrangements. These will provide us with examples of lattices which are not supersolvable but which admit atom decision trees.

A *hyperplane*, $\mathbf{H}$, is an $n-1$ dimensional subspace of the Euclidean space $\mathbf{R}^n$. So all our hyperplanes will go through the origin. An *arrangement* of hyperplanes is a set $\mathscr{A}=\{\mathbf{H}_1, ..., \mathbf{H}_k\}$, where each $\mathbf{H}_i$ is a hyperplane. Let $L=L(\mathscr{A})$ be the poset of intersections of these hyperplanes ordered by reverse inclusion. Thus in L, $\hat{0}$ corresponds to $\mathbf{R}^n$, each atom corresponds to an $\mathbf{H}_i$, and $\hat{1}$ corresponds to $\bigcap_{1\leqslant i\leqslant k}\mathbf{H}_i$. It is well known [15] that L is a semimodular lattice with rank function

$$\operatorname{rk} X = n - \dim X$$

for any $X\in L$. The *characteristic polynomial of* $\mathscr{A}$ is

$$\chi(\mathscr{A}, t) = \sum_{X\in L(\mathscr{A})} \mu(X)\, t^{\dim X}. \tag{6}$$

A comparison of Eqs. (1) and (6) shows that the characteristic polynomials of an arrangement and its associated lattice differ only by a factor of t^k for some k. In all of the examples that we will consider, $k=0$, so we can ignore the difference.

Given any set of vectors in $\mathbf{R}^n$, there is a corresponding hyperplane arrangement gotten by taking the hyperplanes perpendicular to each vector. In particular, there is an arrangement associated with any root system. (For more information about root systems see Humphreys [10].) Let $\mathbf{e}_1, ..., \mathbf{e}_n$ be the standard coordinate vectors in $\mathbf{R}^n$. Then the hyperplane arrangements corresponding to the root systems of types D_n and B_n are

$$\mathscr{D}_n = \{(\mathbf{e}_i \pm \mathbf{e}_j)^\perp : 1\leqslant j<i\leqslant n\}$$

and

$$\mathscr{B}_n = \mathscr{D}_n \cup \{\mathbf{e}_i^\perp : 1\leqslant i\leqslant n\},$$

respectively. Thus we can consider the interpolating arrangements

$$\mathscr{D}\mathscr{B}_{n,k} = \mathscr{D}_n \cup \{\mathbf{e}_i^\perp : 1\leqslant i\leqslant k\}.$$

Note that for $k=0$ or n, $\mathscr{D}\mathscr{B}_{n,k}$ reduces to $\mathscr{D}_n$ or $\mathscr{B}_n$, respectively. Zaslavsky [21] was the first to consider the family of hyperplane arrangements interpolating between $\mathscr{D}_n$ and $\mathscr{B}_n$. These investigations were continued by Cartier [6], Gózefiak and Sagan [12], Orlik and Solomon [14], Orlik *et al.* [11, Example 2.6], Ziegler [24], and Hanlon and Zaslavsky [9].

To describe the lattices for root system arrangements, we will use Zaslavsky's theory of signed graphs [21, 22]. Each element of $L(\mathscr{D}\mathscr{B}_{n,k})$ will be encoded using a graph, G, on the labeled vertex set $[n]=\{1, 2, ..., n\}$. The edges of G will be of three types:

- positive edge between vertices i and j, denoted ij^+,
- a negative edge between vertices i and j, denoted ij^-,
- a half edge with only one endpoint i, denoted i^h.

The edges ij^+, ij^-, and i^h correspond to the roots $\mathbf{e}_i - \mathbf{e}_j$, $\mathbf{e}_i + \mathbf{e}_j$, and $\mathbf{e}_i$, respectively. (In the general theory there are also *loops*, which are edges with two endpoints at the same vertex i, corresponding to the root $2\mathbf{e}_i$.) The reason for the choice of signs will be explained shortly.

To characterize the graphs which appear in $L(\mathscr{DB}_{n,k})$, we need some notation. For any $V \subseteq [n]$, let K_V^+ (respectively, K_V^-) denote the signed complete graph on the vertex set V consisting of all positive (respectively, all negative) edges. Similarly, let $K_{V,W}^+$ ($K_{V,W}^-$) denote the complete bipartite graphs between vertex sets V and W which are all positive (all negative). In using this notation, we tacitly assume that $V \cap W = \varnothing$. Finally, let $K_V^{\pm(k)}$ be the complete signed graph, i.e., the one that has all edges of both signs between vertices in V together with all half edges on $V \cap [k]$.

THEOREM 2.1. *The lattice* $L(\mathscr{DB}_{n,k})$ *is isomorphic to the lattice of subgraphs* G *of* $K_{[n]}^{\pm(k)}$ *such that each component of* G *is of the form*

1. $K_V^+ \cup K_W^+ \cup K_{V,W}^-$, *or*
2. $K_V^{\pm(k)}$.

Furthermore, there can be at most one component of type 2.

If edge e corresponds to root $\mathbf{e}$, then the isomorphism of the preceding theorem is obtained by sending G to $\bigcap_{e \in G} \mathbf{e}^\perp$. The reason for our choice of edge signs is as follows. A *cycle of length* n in G is a sequence of vertices $v_1, v_2, ..., v_n$, where v_i and v_{i+1} are connected by an edge for all i modulo n. We include the cases when $n = 2$ and there is a pair of edges between v_1 and v_2, or when $n = 1$ and there is a half edge at v_1. A cycle is *balanced* if the product of the signs of its edges is positive and *unbalanced* otherwise. A cycle consisting of a half edge will be considered to be unbalanced. Now a component of type 1 can be called balanced (all of its cycles are balanced), while a component of type 2 is unbalanced (at least of its cycles is unbalanced). Be sure to distinguish a cycle from a circuit as defined in Section 1.

Finally, it is convenient to attach a new component, denoted $\varnothing$, to every G which does not have an unbalanced component. The new component will have no vertices or edges and will be considered to be unbalanced. Thus every graph in $L(\mathscr{DB}_{n,k})$ will now have exactly one unbalanced component.

To show that most of the lattices $L(\mathscr{D}\mathscr{B}_{n,k})$ are not supersolvable we will need a result which, while part of the folklore of this subject, does not seem to have been explicitly stated before.

PROPOSITION 2.2. *Let L be a semimodular lattice such that $\chi = \chi(L, t)$ factors over the positive integers. Fix some $r \geqslant 1$ and let R be the sum of the smallest r roots of χ. If every element $x \in L$ of rank r is above less than R atoms, then L is not supersolvable.*

Proof. Suppose, to the contrary, that L is supersolvable with M-chain C. Let x be the element of rank r in M. Then by Theorem 1.2, the number of atoms that x is above must be a sum of r roots of χ. But by assumption, x is above fewer than R atoms, and R is the smallest such sum. This is a contradiction. ∎

The polynomial $\chi(\mathscr{D}\mathscr{B}_{n,k}, t)$ factors, as was noted by Cartier [6, p. 14].

PROPOSITION 2.3. *The roots of $\chi(\mathscr{D}\mathscr{B}_{n,k}, t)$ are*

$$1, 3, ..., 2n-3, n+k-1.$$

We will also need an expression for the rank and number of atoms below a graph G in $L(\mathscr{D}\mathscr{B}_{n,k})$. The following result can be easily deduced from Theorem 2.1.

PROPOSITION 2.4. *Suppose the graph $G \in L(\mathscr{D}\mathscr{B}_{n,k})$ has components on the vertex sets $V_0, V_1, ..., V_l$, where the unique unbalanced component is on the set V_0. Let $v_i = |V_i|$ for all i. Then the rank of G is*

$$\operatorname{rk}(G) = n - l$$

and the number of atoms below G is

$$v_0(v_0-1) + h + \sum_{i \geqslant 1} \binom{v_i}{2},$$

where $h = |V_0 \cap [k]|$.

We can now combine the previous three propositions to determine for which n, k the lattice $L(\mathscr{D}\mathscr{B}_{n,k})$ is not supersolvable.

THEOREM 2.5. *If $k \leqslant n-2$ then $L(\mathscr{D}\mathscr{B}_{n,k})$ is not supersolvable unless $k=0$ and $n \leqslant 3$.*

Proof. First consider the case $k > 0$. Let G be an element at rank $k+1$ in $L(\mathscr{D}\mathscr{B}_{n,k})$. By Proposition 2.4, G will be above a maximum number of atoms if it has a unique nontrivial component on the first $k+1$ vertices which is unbalanced. In this case, G covers

$$(k+1)k+k=k^2+2k$$

atoms.

Now look at the roots of $\chi(\mathscr{D}\mathscr{B}_{n,k}, t)$ as given in Proposition 2.3. If $n \geqslant k+2$ then the root $n+k-1$ is at least $2k+1$. Thus the $k+1$ smallest roots are $1, 3, ..., 2k+1$. But then the sum of these roots is

$$(k+1)^2=k^2+2k+1.$$

So the lattice satisfies the hypotheses of Proposition 2.2 and is thus not supersolvable.

For the case $k=0$, we consider all elements G at rank 2 (since the ones at rank $k+1=1$ are just atoms). Now an element with the largest number of atoms below is $K^+_{[3]}$. The number of such atoms is 3. But if $n \geqslant 4$, then the two smallest roots of $\chi(\mathscr{D}_n, t)$ are 1 and 3. Since $3<1+3$, we are again done by Proposition 2.2. ∎

In the next three sections we will show that every $L(\mathscr{D}\mathscr{B}_{n,k})$ admits an atom decision tree. First, however, we will need more information about the lattice. This is provided in the next section, where we concentrate on the case $k=0$. Then in Sections 4 and 5 we construct the trees.

3. The Lattice $L(\mathscr{D}_n)$

In this section we will characterize the NBC bases of $L(\mathscr{D}_n)$. Along the way we will mention characterizations of its independent sets and circuits as well.

Let G be a graph in the lattice for $\mathscr{D}_n$. Then when we write an edge of G in the form ij^ε for some sign ε, we are assuming that $i>j$. We say that there is a *simple edge* of G between i and j if $ij^+ \in G$ or $ij^- \in G$ but not both. We say that there is a *double edge* of G between i and j if $\{ij^+, ij^-\} \subseteq G$. Recall that a set B of atoms of $L(\mathscr{D}_n)$ corresponds to a set of edges (one for each atom in B) and so can be considered as a graph.

The next two propositions are special cases of [22, Theorem 5.1]. They are also straightforward to prove directly from Theorem 2.1 and Proposition 2.4, so we will content ourselves with stating the results.

PROPOSITION 3.1. *A set B of atoms of $L(\mathscr{D}_n)$ is independent if and only if every component of B has at most one cycle, and that cycle (if it exists) must be unbalanced.*

PROPOSITION 3.2. *A set B of atoms of $L(\mathscr{D}_n)$ is a circuit if and only if*

1. *B is a balanced cycle or*

2. *B contains exactly two cycles, both unbalanced, with a path connecting them.*

Note that in the second case, the path connecting the cycles may have length zero, so that the cycles intersect in precisely one point.

To characterize sets containing broken circuits, we will need a few definitions. First of all we must introduce the atom ordering that we will use. Define the *positive lexicographic ordering* (PLO) of the atoms of $L(\mathscr{D}_n)$ by taking the lexicographic ordering on unsigned edges and then replacing each unsigned edge ij by $ij^+ < ij^-$. So, for example, the PLO for $L(\mathscr{D}_3)$ is

$$21^+ < 21^- < 31^+ < 31^- < 32^+ < 32^-.$$

Next, we will need names for the graphs whose presence indicates a broken circuit. A *camel hump, H,* is a pair of edges ij^ε, ik^δ where $i > j, k$ and $j \neq k$. The hump is so-called because of Fig. 3a, where the relative sizes of i, j, k are indicated by their heights. A *decreasing snake from x_0 to x_k, S,* is a set of edges

$$x_0x_1^+,\ x_0x_1^-,\ x_1x_2^{\varepsilon_2},\ x_2x_3^{\varepsilon_3},\ \ldots,\ x_{k-1}x_k^{\varepsilon_k},$$

where $x_0 > x_1 > \cdots > x_k$. The *mouth* of the snake is the double edge between x_0 and x_1. A snake may simply consist of a mouth with no other edges. Such an animal is depicted in Fig. 3b. An *almost decreasing snake* is like a snake except that we must have ε_k negative and we only insist that

$$x_0 > x_1 > \cdots > x_{k-1} \qquad \text{and} \qquad x_k < x_{k-2}.$$

Thus the *tail* of such a snake is either $x_{k-1}x_k^-$ or $x_kx_{k-1}^-$. An almost decreasing snake with the latter choice for a tail is shown in Fig. 3c. Sometimes we will denote these snakes simply by $x_0, x_1, \ldots, x_k$ if it is clear where the head and tail lie.

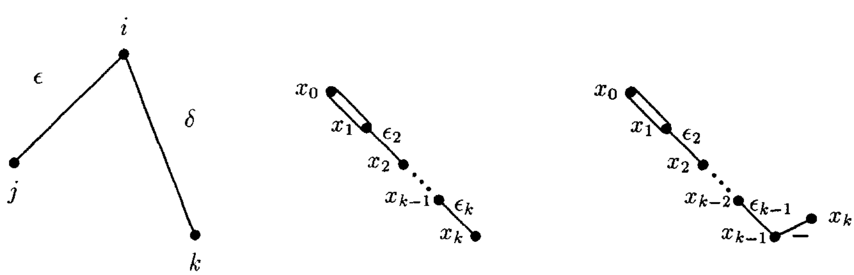

FIG. 3. A menagerie. (a) camel hump; (b) decreasing snake; (c) almost decreasing snake.

PROPOSITION 3.3. *A set B of atoms of $L(\mathscr{D}_n)$ contains a broken circuit under the positive lexicographic ordering if and only if B contains one of the following*:

1. *a camel hump, H,*

2. *two disjoint decreasing snakes S_1 from x_0 to x_k and S_2 from y_0 to y_l, such that $x_{k-1} > y_l > x_k$,*

3. *an almost decreasing snake S.*

Proof. We will keep the notation of the previous definitions and consistently use Proposition 3.2 without mention.

For the "if" direction, it suffices to show that each of the three choices is a broken circuit. It is easy to check that in all three cases we can form a circuit C by adding a smallest edge, e as follows. In case 1, $C = H \cup \{jk^\gamma\}$, where the sign γ is chosen so that the cycle is balanced. In case 2, $C = S_1 \cup S_2 \cup \{y_l x_k^+\}$. And in case 3, $C = S \cup e^+$ where e^+ is the other edge connecting the two vertices of the tail.

For the "only if" direction, suppose B contains a broken circuit. Then this broken circuit comes from removing the smallest edge of one of the graphs G of Proposition 3.2. There are two cases depending upon whether G contains a cycle of length at least three or not.

In the first case, let C be the cycle $x_0, x_1, ..., x_k$, where $k \geqslant 2$. Then $B \supseteq C - \{e\}$, where $e = x_k x_0^\varepsilon$ is C's smallest edge. Now $x_k > x_0$ implies that $x_1 > x_k > x_0$, otherwise the edge connecting x_0 and x_1 would be smaller than e. So the previous sequence of vertices starts off increasing and must eventually decrease to get to x_k. Thus if x_i is the first vertex of this sequence with $x_i > x_{i+1}$, then we have $i \geqslant 1$. Hence $x_i x_{i-1}^\delta, x_i x_{i+1}^\gamma$ is a camel hump in B.

Now suppose the only cycles in G are of length 2. Thus these cycles must be unbalanced. So, by the second condition in Proposition 3.2, $B \supseteq P - \{e\}$, where P is a path with a double edge at each end. First note that if P contains a camel hump, then we are back in the first case. So we may assume that the vertices of P from a *reverse unimodal sequence*: decreasing first and then increasing. Thus removing the least edge e results in either two decreasing snakes as described or an almost decreasing snake, depending upon whether e is a simple or double edge of P, respectively. ∎

We have the following useful corollaries:

COROLLARY 3.4. *Suppose a set B of atoms of $L(\mathscr{D}_n)$ is NBC and pick any vertex $x \in [n]$. Then there is at most one vertex $a < x$ such that*

$$\{xa^+, xa^-\} \cap B \neq \varnothing.$$

Proof. If there are two such vertices, then B contains a camel hump, contradicting the previous proposition. ∎

COROLLARY 3.5. *Let B be an NBC base of $L(\mathscr{D}_n)$ under the positive lexicographic order. Then, the set*

$$B_a = \{xy^\varepsilon \in B \mid x, y \leqslant a\}$$

is an NBC base of $L(\mathscr{D}_n)$ under the positive lexicographic ordering.

Proof. If B does not contain any of the configurations listed in Proposition 3.3, then B_a cannot either. ∎

We are now ready to give an inductive characterization of the *NBC* bases of $L(\mathscr{D}_n)$. It is this theorem that will permit us to construct an ADT. In all that follows, $\uplus$ stands for disjoint union.

THEOREM 3.6. *The set B is an NBC base of $L(\mathscr{D}_n)$ if and only if $B = X \uplus B_m$, where B_m is an NBC base for $L(\mathscr{D}_m)$, and the pair X, m satisfies one of the four following mutually exclusive conditions*:

1. $X = \varnothing, m = n-1.$
2. $X = \{nx^+\}, m = n-1.$
3. $X = \{nx^-\}, m = n-1.$
4. $X \supseteq \{nx_1^+, nx_1^-\}$ *for some x_1, where these are the only edges in B containing n; $m = a-1$, where "a" is the smallest label on a vertex in the connected component of B containing n, and both of the following hold.*

   (a) *There is a decreasing snake in X on a vertex set $n = x_0 > x_1 > \cdots > x_k = a$ with all simple edges positive.*

   (b) *For all $y, n > y > a$, there is at most one edge of the form $yz^\varepsilon \in X$. Moreover, if $x_{i-1} \geqslant y > z = x_i$ for some i, then $\varepsilon = +$.*

Proof. Suppose B is an NBC base for $L(\mathscr{D}_n)$. Then B_m is an NBC base for $L(\mathscr{D}_m)$ by Corollary 3.5. We will now verify that one of the four conditions holds. (It is easy to see that they are mutually exclusive.) We will also say that a base is of *type t* if it satisfies condition t, $t = 1, ..., 4$.

By Corollary 3.4, $(nx^+, nx^-\} \cap B \neq 0$ for at most one $x < n$. So B falls into one of the following four cases:

(i) $B = B_{n-1}$

(ii) $B = \{nx^+\} \uplus B_{n-1}$

(iii) $B = \{nx^-\} \uplus B_{n-1}$

(iv) $B = \{nx^+, nx^-\} \uplus B_{n-1}.$

Clearly, cases (i)–(iii) correspond exactly to bases of types 1–3, respectively. It remains to show that (iv) and 4 correspond.

So let a be the smallest label on a vertex in the connected components of B containing n. Thus there is a path $n = x_0, x_1, ..., x_k = a$. The vertices on the path must decrease, otherwise B contains a camel hump and so is not NBC by Proposition 3.3. Furthermore, all the edges except nx_1^+, nx_1^- on this path must be simple and positive, otherwise B contains an almost decreasing snake, again contradicting Proposition 3.3. This verifies condition 4a.

For 4b, we proceed by contradiction. By Corollary 3.4 the only way to have two edges of the given form is if $\{yz^+, yz^-\} \subseteq X$. We now have two cases, depending on whether $x_{i-1} \geqslant y > z = x_i$ for some i, or not.

If $x_{i-1} \geqslant y > z = x_i$, then the presence of $yz^- = yx_i^-$ will force X to contain an almost decreasing snake, a contradiction to Proposition 3.3. Thus we can only have yz^+, and this proves the "moreover" statement.

The only other possibility is to have $x_{i-1} \geqslant y > x_i$ for some i but $z < x_i$. Then $x_0x_1^+, x_0x_1^-, x_1x_2^+, ..., x_{i-1}x_i^+$ and yz^+, yz^- are a pair of decreasing snakes satisfying the second condition of Proposition 3.3. This contradiction ends the proof of the forward direction of the theorem.

For the reverse implication, suppose B has one of the types 1 through 4. It will suffice to show that B contains no broken circuit (since then B must be a base). We now consider each of the four possibilities in turn.

Type 1. B is an NBC base of $L(\mathscr{D}_{n-1})$. The fact that B is also an NBC base of $L(\mathscr{D}_n)$ follows immediately from the equivalence in Proposition 3.3.

Type 2. $B = \{nx^+\} \cup B_{n-1}$ with B_{n-1} an NBC base of $L(\mathscr{D}_{n-1})$. Suppose, to the contrary, that B contains a broken circuit. Then B contains one of the three graphs G in Proposition 3.3. If we can show in each case that $B_{n-1} \supseteq G$ then this will contradict B_{n-1} being NBC. Equivalently, we must demonstrate that we always have $nx^+ \notin G$. Now vertex n has degree one in B. (The *degree* of a vertex is the number of edges containing it.) But if n were a vertex in any of the three possible G's, then it would have to have degree 2. Thus $nx^+ \notin G$ in all cases.

Type 3. The same argument as for type 2 can be applied, with nx^+ replaced by nx^-.

Type 4. We note that by the arguments for types 2–3 and condition 4b, the sets $B_{a-1}, B_a, ..., B_{n-1}$ are all NBC bases of $L(\mathscr{D}_{n-1})$. Moreover, applying the same arguments, we must have that $B_{n-1} \cup \{nx_1^+\}$ and $B_{n-1} \cup \{nx_1^-\}$ are both NBC bases for $L(\mathscr{D}_n)$.

Now suppose that B is not NBC, and let D be a broken circuit contained in B. From the previous remarks, $\{nx_1^+, nx_1^-\} \subset D$. By Proposition 3.2,

D must come from removing an edge of a path with both ends double. Thus $D = S$ or $D = S_1 \cup S_2$ where these snakes are as described in Proposition 3.3, parts 3 and 2, respectively. Furthermore nx_1^+, nx_1^- must be the mouth of S in the first case, or of one of S_1, S_2 in the second. Note that by the definition of type 4, any path with decreasing vertices that starts at n must be contained in the snake

$$S' : n = x_0, x_1, ..., x_k = a.$$

So in the case $D = S$, S cannot have a down tail because all the simple edges of S' are positive. S cannot have an up tail since that would contradict the second half of condition (b). Thus $D = S$ is impossible.

But if $D = S_1 \cup S_2$ then one of the snakes must be contained in S'. All double edges other than nx_1^+, nx_1^- must lie in B_{a-1} by the first half of (b). Thus all vertices on one snake are $\geqslant a$ while all vertices on the other are $< a$. Thus it is impossible for the ends of S_1 and S_2 to dovetail as they must in the second part of Proposition 3.3. This eliminates the final case, $D = S_1 \cup S_2$, and proves the theorem. ∎

4. The Decision Tree for $L(\mathscr{D}_n)$

To build the tree for $L = L(\mathscr{D}_n)$ we will also need conditional assignment statements for edge labels. Specifically, consider an edge e entering a vertex v of the tree, where v has $2s$ edges leaving it for some s. Then we can give e the label

$$B := \begin{cases} B \cup \{ij^-\} & \text{if the next edge traversed is one of the first } s \text{ leaving } v, \\ B \cup \{ij^+\} & \text{if the next edge traversed is one of the last } s \text{ leaving } v. \end{cases}$$

(Recall that our trees are ordered, so "first" and "last" make sense.) It will simplify things to abbreviate this label to $ij^{\pm}$. After following the next tree edge, we will say that the choice for $ij^{\pm}$ has been *resolved.*

The atom decision tree for the $L(\mathscr{D}_n)$ will be denoted T_n, and it will have height n. For ease of reference, in addition to labeling each edge of the tree we will add a color to the non-root vertices, namely green (G), white (W), red (R), or silver (S). Hence, we will define the tree by stating the labels on the edges and the colors of the respective children. Also, for a given vertex it will be convenient to let t denote its level and $s = n - t$ denote its colevel.

The root has n children; the labels and colors are

$\varnothing$		colored white
$na^{\pm}$	for $a = 1, ..., n-1$	colored green.

A white vertex at level t has $2s$ children with edge labels and vertex colors

$\varnothing$		colored white
$sa^{\pm}$	for $a=1, ..., s-1$	colored green
$(s+1)\,a^{+}$	for $a=1, ..., s$	colored white.

A green vertex at level t has $2s$ children with edge labels and colors of children given by

$\varnothing$		colored white
$sa^{\pm}$	for $a=1, ..., s-1$	colored green
$(s+1)\,s^{-}$		colored red or silver
$sa^{\pm}$	for $a=1, ..., s-1$	colored green.

To decide on the choice of red or silver, we make the following observation, which will be verified after completing the definition of the tree. Note that when we reach a vertex v of T_n then the set of (resolved) elements on edges from the root to v form a graph G_v. If v is at level t, then there will be at most one component of G_v containing both a double edge and a vertex with label less than or equal to $s+1$. Furthermore, this vertex will be unique (if it exists), so let its label be l. Finally, if v is to be colored red or silver then such a component must exist, so we can choose

$$\text{color} = \begin{cases} \text{red} & \text{if } \; l=s+1 \\ \text{silver} & \text{if } \; l<s+1. \end{cases}$$

A red vertex at level t (and hence $l=s+1$) has $2s$ children with edge labels and vertex colors

$\varnothing$		colored white
$sa^{\pm}$	for $a=1, ..., s-1$	colored green
$(s+1)\,a^{+}$	for $a=1, ..., s$	colored red or silver,

where the red color is chosen if and only if $a=s$.

A silver vertex at level t (with $l<s+1$) has $2s$ children with edge labels and vertex colors

$\varnothing$		colored red or silver
$(s+1)\,a^{-}$	for $a=1, ..., s, a\neq l$	colored red or silver
$(s+1)\,a^{+}$	for $a=1, ..., s$	colored red or silver,

where the red color is chosen if and only if $l=s$. As an example, the tree T_3 is displayed in Fig. 4.

To see that T_n is well defined, we must make several observations. First of all, we need to see that all edge labels of the form $ij^{\pm}$ are resolved by the time we reach a leaf. Note that an edge with such a label will always lead to a green vertex, and the choice is always resolved by any edge leaving that vertex. So we only need to check that no leaf is colored green. That is guaranteed by the next lemma, which is easily checked by inspection.

LEMMA 4.1. *Suppose i is given $1<i\leqslant n$. Then elements of the form ij^{ε} can only label edges leaving level $n-i+1$. Also, elements of the form $ij^{\pm}$ can only label edges leaving level $n-i$.*

We must also check our method of determining whether a vertex is red or silver. For the purposes of this proof it will be convenient to call a component *double* if it contains a double edge.

LEMMA 4.2. *If v is at level t in T_n, then there is at most one double component of G_v containing a vertex with label less than or equal to $n-t+1$. The vertex itself, if it exists, is unique. Finally*

$$\textit{such a component does not exist} \Leftrightarrow v \textit{ is to receive color } W \textit{ or } G,$$

equivalently

$$\textit{such a component exists} \Leftrightarrow v \textit{ is to receive color } R \textit{ or } S.$$

Proof. We induct on the level t. The lemma is readily checked for $t\leqslant 1$. Suppose the result holds for levels up through t and that w is a vertex at level $t+1$. Let v be the parent of w. By the induction hypothesis, all double components of G_v contain only vertices $\geqslant n-t+2$, except for one called C. (We allow $C=\varnothing$ to cover the case when such a component does not exist.) But all of the edges ij^{ε} that can be added in passing from v to w have $i\leqslant n-t+1$ by Lemma 4.1. So all of the double components of G_v, except

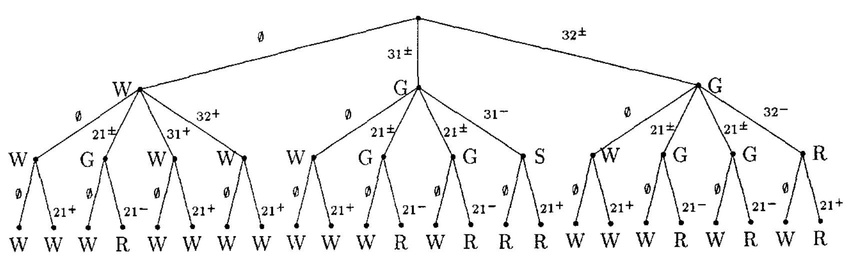

FIG. 4. The tree T_3.

possibly C, remain unchanged in G_w. Thus we only need to worry about what happens to C. There are several cases depending on the color of v.

Suppose first that v is W or G. Then $C = \varnothing$ by induction. Now a quick check of the definition of T_n shows that a double component (actually a double edge) with a vertex $\leqslant n-t$ will be created if and only if w is R or S. Thus the lemma is true in this case.

Now suppose v is R. Thus C's smallest vertex is labeled $n-t+1$ by induction and the definition of red vertices. If vw comes from the last choice in the definition of a red vertex, then C is extended to a vertex $a \leqslant n-t$ by the edge $(n-t+1)\,a^+$, and w is R or S. In the other two choices, C remains the same in passing from G_v to G_w and no new double edge is added. Thus no double component with vertices $\leqslant n-t$ exists in G_w and w is W or G. Thus the lemma holds in this case, too.

Finally, suppose v is S. Now C's smallest vertex is labeled $<n-t+1$. But in all three choices listed for silver vertices, at most a simple edge is added, which cannot intersect a double component other than C. Thus C still exists as the unique double component with vertices $\leqslant n-t$ and w must be colored R or S. This successfully concludes the last case, so we are done with the proof of the lemma. ∎

We are finally in a position to show that D_n admits T_n as an ADT tree. It is clear that T_n satisfies condition 1 of the definition of an ADT. It remains to check condition 2. To do this we define a *branch* as a rooted tree whose root has degree one. In any rooted tree, an edge leaving v and entering w has a corresponding branch consisting of vw and the subtree rooted at w. Also, any tree isomorphisms mentioned in the proof of the following theorem will also preserve vertex colors and (to a certain extent) edge labels.

THEOREM 4.3. *Paths from the root, r, to a leaf in the decision tree T_n are in one-to-one correspondence with the NBC bases of $L(\mathscr{D}_n)$ under the positive lexicographic ordering.*

Proof. We proceed by induction on n. For the case where $n=2$, the result is easily verified. Assume the result for T_m for all values of $m<n$. It suffices to show that paths in T_n correspond to the bases of types 1–4 in Theorem 3.6.

To begin, consider the white child of the root, v_W. Remove all branches starting with edges leaving v_W of the form na^+. What is left of the subtree at v_W is isomorphic to T_{n-1}, by construction. Moreover, since the label of the edge rv_W is $\varnothing$, this part of the subtree will produce all bases of type 1 exactly once, by induction.

For the bases of type 2, pick any $x<n$ and consider the vertex, v_G, entered by leaving the root along the edge labeled $nx^{\pm}$. Take the branches

starting with edges of the form nx^+, $(n-1)\,a^{\pm}$ leaving v_{G} together with the branch starting with nx^+ leaving v_{W}, and paste their roots together. The resulting tree will again be isomorphic to T_{n-1}. The only difference in the edge labeling is that every edge leaving the root will also contain the label nx^+, indicating that these branches will yield all type 2 bases exactly once.

The bases of type 3 are handled in the same way as those of type 2 with nx^+ replaced everywhere with nx^-. The only difference is that, instead of a branch from v_{W}, one uses the branch leaving v_{G} labeled nx^-, $\varnothing$.

For the final stage, we must show that the paths in T_n not using any edges in subtrees of previous cases correspond to type 4 bases. Note that such paths are exactly those that follow an edge $nx^{\pm}$ to a green vertex and then leave it by an edge nx^+, nx^-. Thus if B is the set of elements corresponding to a given path, P, then $\{nx^+, nx^-\} \subseteq B$.

We will first show that B is of type 4, and thus an NBC base of $\mathscr{D}_n$. So let a be the smallest label on a vertex in the component C of B containing n. Because of Lemma 4.2, all vertices $v_2, v_3, \ldots, v_{n-a+1}$ on P from levels 2 to $t=n-a+1$ must be R or S. Any tree edge between two vertices which are both R or S must be labeled with a single element. This covers the first half of 4b. Now v_t, $2 \leqslant t < n-x+1$, must be silver since at this point of the construction C's smallest vertex is x (because of the snake nx^+, nx^-). Thus, by the second choice for S vertices, we can never have an edge yx^- in C. Furthermore, when we reach v_{n-x+1} it will be red, by definition. The only way to leave this vertex for another R or S vertex is to choose a positive edge xx_2^+, which will extend our snake to $n > x > x_2$. Iterating this argument shows that 4a and the second half of 4b hold.

To complete this part of the proof, we must examine B_{a-1}. By the arguments in the previous paragraph, v_{n-a+1} must be red. It follows that P must continue to a W or G vertex at level $n-a+2$, since otherwise C would be extended to a smaller vertex. But the branches corresponding to the edges available to P at this point from a tree isomorphic to T_{a-1}, so B_{a-1} is a base of $\mathscr{D}_{a-1}$ by induction.

Last, we must show that given any NBC base B of type 4, there is a unique path corresponding to B. By definition

$$B = \{nx^+, nx^-\} \bigcup_{y=a+1}^{n-1} A_y \cup B_{a-1}, \tag{7}$$

where a is the smallest vertex label in the component of the graph of B containing n, and $A_y = \{yz^{\varepsilon}\}$ if an edge of the form $yz^{\varepsilon} \in B$, and $A_y = \varnothing$ otherwise. In this case we make the initial choices of edge labels

$$nx^{\pm};\ \{nx^+, nx^-\};\ A_{n-1};\ \ldots;\ A_{a+1}$$

ending at a vertex v on level $n-a+1$. Then, as has been previously noted, the tree T_{a-1} is isomorphic to the subtree consisting of all branches from v to a W or G vertex. Therefore, by induction, there exists a continuation of the initial path giving the NBC base B. Since the expression (7) is unique, the initial part of the path is uniquely determined. By induction, the terminal portion is also unique. This concludes the proof of the theorem. ∎

Thus the tree T_n gives a combinatorial explanation for the factorization of the characteristic polynomial of $L(\mathscr{D}_n)$.

5. The Decision Tree for $L(\mathscr{D}\mathscr{B}_{n,k})$

To handle the $L(\mathscr{D}\mathscr{B}_{n,k})$ case, we will first restate some of the propositions from Section 3 in this setting. The proofs are similar to those previously given and so will be omitted. Note that Propositions 3.1–3.2 continue to hold, remembering that a half edge is considered to be an unbalanced cycle.

To characterize the broken circuits, we need to extend the PLO ordering. First order all the half edges lexicographically, and make them all greater than every edge of the form ij^{ε}. By way of illustration, the ordering for $L(\mathscr{D}\mathscr{B}_{3,2})$ is

$$21^+ < 21^- < 31^+ < 31^- < 32^+ < 32^- < 1^h < 2^h < 3^h.$$

We also need to define the analogs of our various snakes. A *decreasing worm from* x_0 to x_m is a set of edges

$$x_0^h, x_0x_1^{\varepsilon_1}, x_1x_2^{\varepsilon_2}, ..., x_{m-1}, x_m^{\varepsilon_m},$$

where $x_0 > x_1 > \cdots > x_m$. An *almost decreasing worm* has a negative tail which can be either increasing or decreasing. A *reptile* is a graph which is either a snake or a worm.

Proposition 5.1. *A set B of atoms of $L(\mathscr{D}\mathscr{B}_{n,k})$ contains a broken circuit under the positive lexicographic ordering if and only if B contains one of the following*:

1. *a camel hump, H,*

2. *a decreasing snake, $x_0 > \cdots > x_m$, disjoint from a decreasing reptile, $y_0 > \cdots > y_l$, such that $x_{m-1} > y_l > x_m$,*

3. *a decreasing worm* $x_0 > \cdots > x_m$, *disjoint from a decreasing reptile,* $y_0 > \cdots > y_l$, *such that* $x_m < y_0$,

4. *an almost decreasing reptile.*

Rather than stating an analog of Theorem 3.6, we will prove a proposition that will characterize the NBC bases of $L(\mathscr{D}\mathscr{B}_{n,k})$ in terms of those of $L(\mathscr{D}_n)$ and $L(\mathscr{D}_{n+1})$.

PROPOSITION 5.2. *Any NBC B base of* $L(\mathscr{D}\mathscr{B}_{n,k})$ *contains at most one half edge. Moreover,*

1. *B contains no half edge if and only if B is an NBC base of* $L(\mathscr{D}_n)$,

2. $B = \{x^h\} \cup B'$ *if and only if* B' *is a set of atoms of* $L(\mathscr{D}_n)$ *and* $B'' = B' \cup \{(n+1)\,x^+, (n+1)\,x^-\}$ *is an NBC base of* $L(\mathscr{D}_{n+1})$.

Proof. By condition 3 of Proposition 5.1, B cannot contain two or more half edges. Furthermore, if we eliminate the possible graphs with half edges from the list in that proposition, then we get the list in Proposition 3.3. This proves the first equivalence in the theorem.

The proofs of the forward and reverse implications of the second equivalence are very similar, so we will only do the former. So suppose $B = \{x^h\} \cup B'$ is NBC. Thus, as already noted, B' contains no half edges. So B' is a set of atoms in $L(\mathscr{D}_n)$. To show that B'' is NBC, we proceed by contradiction.

If B'' contains a broken circuit, C, then we go through the possibilities in Proposition 3.3 one at a time. First note that if $C \subseteq B'$ then $C \subseteq B$, contradicting Proposition 5.1. Thus we can assume that $\{(n+1)\,x^+, (n+1)\,x^-\} \cap C \neq \varnothing$. This rules out the camel hump, since these are the only edges of B'' containing $n+1$. If C is a pair of decreasing interlocking snakes, then $\{(n+1)\,x^+, (n+1)\,x^-\}$ must be the mouth of one of them. Hence replacing that pair with x^h will give either configuration 2 or 3 from the previous proposition, a contradiction. The same replacement works if C is an almost decreasing snake, turning it into an almost decreasing worm. This last contradiction exhausts the possibilities for C and finishes the proof. ∎

Finally, we define the tree $T_{n,k}$ as follows. Take the tree T_n and add k new edges leaving the root given as follows:

$$j^h \qquad \text{for} \quad j = 1, ..., k \text{ colored red } (k = n) \text{ or silver } (k \neq n).$$

We then let all succeeding vertices have descendents as before. However, the general choice of whether a vertex is red or silver depends on the unique component with either a double or half edge that contains a label

$l \leqslant n-t+1$. The choice being R or S if $l=n-t+1$ or $l<n-t+1$, respectively. Again, the first two conditions for a modified ADT are clearly satisfied. The next theorem takes care of the third.

THEOREM 5.3. *Paths from the root, r, to a leaf in the decision tree* $T_{n,k}$ *are in one-to-one correspondence with the NBC bases of* $L(\mathscr{D}\mathscr{B}_{n,k})$ *under the positive lexicographic ordering.*

Proof. By Proposition 5.2, the NBC bases of $L(\mathscr{D}\mathscr{B}_{n,k})$ are of two types: those that contain no half edge and those that contain exactly one half edge. By Theorem 4.3, those of the first type are each given exactly once by the part of $T_{n,k}$ which is isomorphic to T_n.

We must also show that those bases containing a single half edge are in one-to-one correspondence with paths in branches added to T_n to form $T_{n,k}$. But by construction, the branch starting with the edge labeled x^h is isomorphic to the branch of T_{n+1} corresponding to the edge labeled $(n+1)\,x^+, (n+1)\,x^-$. By Theorem 4.3 and Proposition 5.2, again, the bases and paths match up. ∎

6. COMMENTS AND OPEN QUESTIONS

We hope that this paper is only the first to work on this topic. The following is a list of possible avenues for future research.

(1) It would be simpler if the only assignment statements need to label the edges of our ADTs were of the form $B := B \cup \{a\}$. Unfortunately, we were only able to construct such trees for $\mathscr{D}_n$ when $n \leqslant 4$. (Note that this includes the first non-supersolvable case.) It is possible for general n?

It would be interesting to extend this method to other non-supersolvable lattices such as those for the exceptional roots systems, lattices associated with certain free arrangements of hyperplanes [20, 11, 12], the generalized Dowling lattices [8], certain lattices associated with partitions into even and odd block size [5], and frame matroid lattices [23]. Since some of these posets are not semimodular lattices, an appropriate analog of Theorem 1.1 will also have to be found for such cases.

(2) The reader may be wondering how to construct an ADT, T, for a given lattice, L. The following method, while lengthy, may work. First, list all the NBC bases of L. Next, choose a subset $A=\{a_1, a_2, ..., a_r\}$ of the atoms of L, where r is a root of $\chi(L, t)$. Now construct, if possible, a partition of the NBC bases into subset $S_1, S_2, ..., S_{r+1}$ so that a_i is a member of all bases in S_i for $1 \leqslant i \leqslant r$. (It is convenient, but not necessary, if every base in S_{r+1} is disjoint from A.) This construction is now iterated, using

a new root of the characteristic polynomial, and using the NBC bases in S_i to find the labeling of the edges leaving the corresponding vertex. If the initial set of atoms does not work, then try a different set (possibly with the cardinality of a different root of $\chi(L, t)$).

It would be nice to have a construction process that more closely mirrors the way an atom partition is found in the supersolvable case, using an M-chain. Perhaps there is some small set of elements at each rank of L such that every element of L forms a modular pair with at least one of the elements of this set. In addition, there should be some nice condition on the covering relations between elements of these sets in adjacent ranks.

(3) There are other ways to explain the factorization of characteristic polynomials, $\chi(\mathscr{A}, t)$, for a hyperplane arrangement. In particular, one can associate a module of derivations with the arrangement. If this module is free, then the degrees $d_1, ..., d_n$ in a homogeneous basis is an invariant of the arrangement. In this case, the roots of the characteristic polynomial are just $d_1+1, ..., d_n+1$. Thus, finding a homogeneous basis is a way to algebraically explain the factorization of $\chi(\mathscr{A}, t)$. This technique is used for various subarrangements of Coxeter arrangements, including $\mathscr{D}\mathscr{B}_{n,k}$, in the paper of Józefiak and Sagan [12].

(4) Hélène Barcelo and Alain Goupil [1] have independently come up with a factorization of the NBC complex of $L(\mathscr{D}_n)$ which is similar to ours. Their paper also contains a nice result (Theorem 3.1, joint with Garsia) that relates the NBC sets with reduced decompositions into reflections of Coxeter group elements.

Acknowledgments

We thank Adriano Garsia for suggesting to us the problem of finding a factorization of the NBC complex which explains the factorization of $\chi(\mathscr{D}_n, t)$. We also thank Anders Björner, Hiroaki Terao, and Tom Zaslavsky for their encouragement and stimulating conversations.

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