GCD MATRICES, POSETS, AND NONINTERSECTING PATHS

ERCAN ALTINISIK, BRUCE E. SAGAN, AND NAIM TUGLU

Abstract. We show that with any finite partially ordered set P (which need not be a lattice) one can associate a matrix whose determinant factors nicely. This was also noted by D. A. Smith, although his proof uses manipulations in the incidence algebra of P while ours is combinatorial, using nonintersecting paths in a digraph. As corollaries, we obtain new proofs for and generalizations of a number of results in the literature about GCD matrices and their relatives.

1. Introduction

Let $\mathbb{P}$ denote the positive integers and suppose we are given a subset $S = \{a_1, \ldots, a_n\}$ of $\mathbb{P}$. The corresponding *GCD matrix* is $(S) = (s_{ij})$ where $s_{ij} = (a_i, a_j)$, the greatest common divisor of a_i and a_j. We say that S is *factor closed* if given any $a_i \in S$ and any divisor $d|a_i$ then $d \in S$. H. J. S. Smith [27] proved the following beautiful result about the determinant of (S).

Theorem 1.1 (Smith). *If $S = \{a_1, \ldots, a_n\}$ is factor closed then*

$$\det(S) = \phi(a_1) \cdots \phi(a_n) \tag{1}$$

where ϕ is Euler's totient function. □

Since Smith's pioneering paper, a host of related results have appeared in the literature. For a survey with references, see the paper of Haukkanen, Wang, and Sillanpää [15]. We will show that many of these are special cases of a general determinantal identity, see Theorem 2.2 below, associated with any finite partially ordered set P even if it is not a lattice. This theorem can be proved by appealing to a result of D. A. Smith [26, Section 3, Corollary 2]. But Smith's approach relies on manipulations in the incidence algebra of P while we choose to give a combinatorial proof based on counting families of nonintersecting paths in digraphs. This technique is due to Lindström [22] who was motivated by a problem in matroid theory. Related determinantal identities had been studied earlier by Slater in physics [25] as well as Karlin and McGregor working in a probabilistic context [17]. Later, Gessel and Viennot [11, 12] showed how the method could be applied to a whole host of combinatorial problems. More history can be found in the footnotes of a paper of Krattenthaler [19, pages 9–10].

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The rest of this article is structured as follows. The next section will be devoted to proving our main theorem and giving some preliminary applications. In Section 3, we specialize to the case where P is a meet semilattice. This permits us to invert the sums appearing in the general case. Section 4 investigates what happens to the determinant if the set S is lower closed (the partially ordered set generalization of being factor closed) or meet closed. We end with a section containing comments and an open question.

2. The main theorem

We will first review the digraph machinery which we will need to prove our main theorem. Explanations of any undefined terms from graph theory can be found in the texts of Harary [13] or Chartrand and Lesniak [6]. For definitions of concepts about posets (partially ordered sets), the reader can consult Stanley's book [28].

Let D be a finite digraph with vertices V and arcs A. Let R be a commutative ring with identity and suppose we are given a function (weighting) $\omega : A \rightarrow R$. Then any directed path $p:\ v_0v_1\ldots v_k$ has an associated weight

$$\omega(p)=\prod_{i=1}^{k}\omega(v_{i-1}v_i).$$

We also let

$$\omega(v_0,v_k)=\sum_{p}\omega(p)$$

where the sum is over all directed paths p from v_0 to v_k.

Now suppose we are given two disjoint sets of vertices $V'=\{v_1',\ldots,v_n'\}$ and $V''=\{v_1'',\ldots,v_n''\}$ in V. Consider an n-tuple of directed paths $\pi=(p_1,\ldots,p_n)$ where p_i goes from v_i' to v_i'' for all i, $1\le i\le n$. Then we assign weights to π and to the pair (V',V'') in a way analogous to the one used in the preceding paragraph

$$\omega(\pi)=\prod_{i=1}^{n}\omega(p_i)\quad\text{and}\quad\omega(V',V'')=\sum_{\pi}\omega(\pi)$$

where the sum is over all π where no two of the paths in the n-tuple intersect.

Finally, given a permutation g in the symmetric group S_n, we define $\pi_g=(p_1,\ldots,p_n)$ to be an n-tuple of directed paths such that p_i goes from v_i' to $v_{g(i)}''$ for all i. So the n-tuples considered in the previous paragraph would have $g=e$ where e is the identity permutation. If any pair of paths in π_g intersect then we call the n-tuple *intersecting* and *nonintersecting* otherwise. We can now state a corollary of Lindström's Lemma [22] about enumerating nonintersecting paths.

Lemma 2.1 (Lindström)**.** *Let D be a finite digraph with disjoint vertex sets $V'=\{v_1',\ldots,v_n'\}$ and $V''=\{v_1'',\ldots,v_n''\}$ such that any π_g is intersecting if $g\neq e$. Let $(D)=(d_{ij})$ be the matrix with*

$$d_{ij}=\omega(v_i',v_j'').$$

Then

$$\det(D) = \omega(V', V''). \quad \square$$

Now let P be a finite poset and consider the *incidence algebra* $I(P, R)$ of P over R which consists of all functions $F : P \times P \to R$ such that $F(a, b) = 0$ unless $a \leq b$. The identity element of $I(P, R)$ is the Kronecker delta function

$$\delta(a, b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$$

We will also need the *zeta function* of $I(P, R)$ defined by

$$\zeta(a, b) = \begin{cases} 1 & \text{if } a \leq b, \\ 0 & \text{otherwise.} \end{cases}$$

The zeta function is invertible and its inverse is called the Möbius function μ of $I(P, R)$. In other words, μ is the unique function in $I(P, R)$ satisfying

$$\sum_{a \leq c \leq b} \mu(a, c) = \sum_{a \leq c \leq b} \mu(c, b) = \delta(a, b).$$

Now fix a linear ordering of P which can be used to index the rows and columns of a matrix M. Since we will be taking determinants, it will not matter which linear order is used and we will merely say that the matrix is indexed by P. We can now state and prove our main theorem.

Theorem 2.2. *Let P be a finite poset and let $F, G \in I(P, R)$. Let $(P)_{FG}$ be the matrix indexed by P with entries*

$$p_{ab} = \sum_{c \in P} F(c, a) G(c, b). \tag{2}$$

Then

$$\det(P)_{FG} = \prod_{a \in P} F(a, a) G(a, a).$$

Proof. Construct a digraph D as follows. For the vertices of D, take three copies of the elements of P which we denote by P', P'', and P'''. Now put an arc from $a' \in P'$ to $c''' \in P'''$ if and only if $a \geq c$ in P. Dually, put an arc from $c''' \in P'''$ to $b'' \in P''$ if and only if $c \leq b$. Finally, give weights to the arcs by

$$\omega(a', c''') = F(c, a) \quad \text{and} \quad \omega(c''', b'') = G(c, b).$$

Consider paths p from $a' \in P'$ to $b'' \in P''$. Clearly all such paths have the form $p : a', c''', b''$ where $c \leq a$ and $c \leq b$ in P. So if we take $V' = P'$ and $V'' = P''$ then we have $(D) = (P)_{FG}$ since

$$d_{a'b''} = \omega(a', b'') = \sum_{c \leq a,\ c \leq b} F(c, a) G(c, b) = p_{ab}.$$

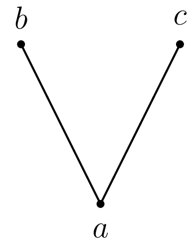

FIGURE 1. A poset P

The last step follows from the fact that a term in p_{ab} corresponding to some c not satisfying the given inequalities is zero.

We need to show that D satisfies the hypotheses of Lemma 2.1. Now take any n-tuple of paths π_g which is nonintersecting. We will show that this forces $g = e$. First we claim that the path starting at any $a' \in P'$ must then follow the arc to $a''' \in P'''$. Suppose this is not true for some a' and let c''' be the next vertex on the path. Then we must have $a > c$ in P. Now consider the path of π_g starting at c'. Since it cannot intersect the previous path, it must go to some d''' with $c > d$. Continuing in this fashion, we can construct an infinite decreasing chain in P, contradicting the assumption that P is finite. So our claim is true. By a dual argument, one can show that the path starting at a' must continue from a''' to a'' and so $g = e$. Furthermore, we have shown that this family π_e is the only nonintersecting path family. So, by the way we have defined the weights and Lemma 2.1,

$$\det(D) = \omega(\pi_e) = \prod_{a \in P} F(a,a)G(a,a). \quad \square$$

Note that one can directly factor the matrix for P as $(P)_{FG} = M_F^t M_G$ where M_F and M_G are the matrices corresponding to the incidence algebra elements F and G, respectively, and t denotes transpose. So one can give a simple linear algebraic proof of the previous theorem just by using the fact that the determinant of a product of matricies is the product of the determinants. However, we prefer the approach given since it combinatorially explains the factorization.

As an example of Theorem 2.2, consider the poset P whose Hasse diagram is shown in Figure 1. Then using the linear order a, b, c one obtains

$$\det \begin{pmatrix} F(a,a)G(a,a) & F(a,a)G(a,b) & F(a,a)G(a,c) \\ F(a,b)G(a,a) & F(a,b)G(a,b) + F(b,b)G(b,b) & F(a,b)G(a,c) \\ F(a,c)G(a,a) & F(a,c)G(a,b) & F(a,c)G(a,c) + F(c,c)G(c,c) \end{pmatrix}$$

$$= F(a,a)G(a,a)F(b,b)G(b,b)F(c,c)G(c,c)$$

As a first application, we note a corollary of Theorem 2.2 which simultaneously generalizes a theorem of Apostol [1] and one of Daniloff [10]. Let f, g be arbitrary functions from the poset P to the ring R. Then on substituting $F(a,b)f(a)$ and $G(a,b)g(a)$ for $F(a,b)$ and $G(a,b)$, respectively, we immediately have the following result.

Corollary 2.3. *Let P be a finite poset. Let $(\overline{P})$ be the matrix indexed by P with entries*

$$\overline{p}_{ab} = \sum_{c \in P} F(c,a)f(c)G(c,b)g(c).$$

Then

$$\det(\overline{P}) = \prod_{a \in P} F(a,a)f(a)G(a,a)g(a). \quad \square$$

Now consider the poset defined by using the divisor ordering on $P_n \stackrel{\text{def}}{=} \{1,2,\ldots,n\}$. Apostol's theorem is obtained by specializing the previous result to the case where $P = P_n$, $F(a,b) = \zeta(a,b)$, $G(a,b) = G(b/a)$ for any function $G : P_n \to R$, and $g(a) = 1$ for all $a \in P$. In this case the determinant becomes

$$\det(\overline{P}_n) = f(1)f(2)\cdots f(n)G(1)^n.$$

With this evaluation in hand, Apostol showed that letting $f(a) = a$ for $a \in P_n$ and $G(a,b) = \mu(b/a)$ where μ is the usual number-theoretic Möbius function gives

$$\det(c(a,b)) = n!$$

where $c(a,b)$ is Ramanujan's sum.

To obtain Daniloff's theorem, let $P = P_n$, $g(a) = 1$ for all $a \in P_n$, and $F(a,b) = G(a,b) = \Omega_k(b/a)$ where

$$\Omega_k(a) = \begin{cases} a^{1/k} & \text{if } a^{1/k} \in \mathbb{P}, \\ 0 & \text{else.} \end{cases}$$

Now Corollary 2.3 becomes Daniloff's result,

$$\det(\overline{P}_n) = f(1)f(2)\cdots f(n).$$

Note that one obtains the same evaluation not just for Ω_k, but also for any functions $F, G : P \to R$ such that $F(a) = G(a) = 1$ for all $a \in P$.

3. Meet semilattices

Now suppose that our poset is a meet semilattice L so that every pair of elements $a, b \in L$ have a greatest lower bound or *meet* $a \wedge b$. Note that in this case the sum (2) can be restricted to $c \le a \wedge b$. This special case of Theorem 2.2 was discovered by Haukkanen, Wang, and Sillanpää [15]. They also showed that by further specialization one could obtain a theorem of Jager [16] concerning a unitary analogue of (1), Smith's evaluation [27] of the LCM determinant, as well as a number of other results.

It would be nice to be able to compute the value of determinants where each entry is a single term, as in Smith's original case, rather than a sum. Since (2) now consists of a sum over all c below a certain element in L, one can use Möbius inversion to accomplish this. Thus we can prove the following theorem of Lindström [21] as a corollary to Theorem 2.2.

Theorem 3.1 (Lindström). *Let L be a finite meet semilattice and let $f \in I(L, R)$. Let $(L)_f$ be the matrix indexed by L with entries*

$$l_{ab} = f(a \wedge b, a)$$

Then

$$\det(L)_f = \prod_{a\in L} \left(\sum_{c\in L} \mu(c,a) f(c,a) \right). \tag{3}$$

Proof. Define a function $F \in I(L, R)$ by

$$F(a,b) = \sum_{c\le a} \mu(c,a) f(c,b)$$

if $a \le b$ and $F(a,b) = 0$ otherwise. By Möbius inversion, this is equivalent to

$$f(a,b) = \sum_{c\le a} F(c,b).$$

It follows that

$$l_{ab} = f(a\wedge b, a) = \sum_{c\le a\wedge b} F(c,a) = \sum_{c\in L} F(c,a)\zeta(c,b).$$

So applying Theorem 2.2 we obtain

$$\det(L)_f = \prod_{a\in L} F(a,a)\zeta(a,a) = \prod_{a\in L} \left(\sum_{c\in L} \mu(c,a) f(c,a) \right). \quad \square$$

This result can also be found as an exercise in Stanley's book [28, Chapter 3, Exercise 37]. The special case where $f(a,b)$ depends only on a was proved independently by Wilf [32]. Smith himself [27] noted that this theorem holds under the further assumption that L is a factor closed subset of $\mathbb{P}$.

To see how Smith's determinant (1) follows from Theorem 3.1, just let $f(a,b) = a$ for all $a \in S$. Then $s_{ij} = f(a_i \wedge a_j, a_i)$ and, since S is factor closed,

$$\sum_{c\in S} \mu(c,a) f(c,a) = \sum_{c|a} \mu(a/c)\ c = \phi(a).$$

4. Lower-closed and meet-closed sets

In work on analogues of Theorem 1.1, there are two conditions that are commonly imposed on the set S. If $S \subseteq P$ for some poset P, we say that S is *lower closed* or a *lower order ideal* if $a \in S$ and $b \le a$ in P implies $b \in S$. This corresponds exactly to S being factor closed if it is a subset of $\mathbb{P}$ ordered by division. If, in particular, our poset is a meet semilattice L and S is lower closed, then S is also a meet semilattice with the same Möbius function as L. So our previous results cover this case without change.

If $S \subseteq L$ for a meet semilattice one can also talk about S being *meet closed* which means that if $a, b \in S$ then $a \wedge b \in S$ where the meet is taken in L. Again, S is also a meet semilattice and so Theorem 3.1 still applies and in fact generalizes a result of Bhat [4] who considered the case when f is a function of only one argument.

However, if $S \subseteq L$ is meet closed we will show that there is also an analogue of (3) which involves the Möbius function of L, rather than that of S. This result will generalize a theorem of Haukkanen [14] who, as in Bhat's paper, only looked at functions of a single variable. Specializing yet further to $L = P_n$ and the function $f(a) = a$ for all $a \in P_n$, one obtains a theorem of Beslin and Ligh [2].

To state our result, it will be convenient to have some notation. Let ℓ be a linear extension of the partial order on S so that if $\ell = a_1, a_2, \ldots, a_n$ then $a_i < a_j$ implies $i < j$. If $d \in L$ then we will write $d \trianglelefteq a_i$ if $d \leq a_i$ and $d \not\leq a_j$ for any $j < i$.

Theorem 4.1. *Let L be a finite meet semilattice and let $f \in I(L, R)$. Suppose $S \subseteq L$ is meet closed and fix a linear extension $\ell = a_1, a_2, \ldots, a_n$ of S. Then*

$$\det(S)_f = \prod_{i=1}^{n} \left(\sum_{d \trianglelefteq a_i} \sum_{c \in L} \mu(c, d) f(c, a_i) \right)$$

Proof. We will use unsubscripted variables for elements of L which are not necessarily in S. Define $F \in I(L, R)$ as in the proof of Theorem 3.1 so that we have

$$f(a_i, a_j) = \sum_{d \leq a_i} F(d, a_j). \tag{4}$$

Also define $\hat{F} \in I(S, R)$ by

$$\hat{F}(a_i, a_j) = \sum_{d \trianglelefteq a_i} \sum_{c \in L} \mu(c, d) f(c, a_j) = \sum_{d \trianglelefteq a_i} F(d, a_j)$$

if $a_i \leq a_j$ and $\hat{F}(a_i, a_j) = 0$ otherwise. If we can show that

$$f(a_i, a_j) = \sum_{a_k \leq a_i} \hat{F}(a_k, a_j)$$

then the rest of the proof will follow as in the demonstration of Theorem 3.1. So, by the definition of $\hat{F}$, it suffices to show

$$f(a_i, a_j) = \sum_{a_k \leq a_i} \sum_{d \trianglelefteq a_k} F(d, a_j). \tag{5}$$

We will do this by showing that there is a one-to-one correspondence between the terms in (5) and those in (4).

First note that each $d \in L$ occurs at most once in (4) and at most once in (5) (since $d \trianglelefteq a_k$ for at most one $a_k \in S$). If d occurs in (5) then $d \leq a_k \leq a_i$ and so d occurs in (4). Conversely, if d occurs in (4) then $d \leq a_i$ and so we must have $d \trianglelefteq a_k$ for some a_k with $k \leq i$. But now $d \leq a_i \wedge a_k = a_l$ for some l since S is meet closed. Furthermore, $a_l \leq a_k$ implies $l \leq k$, and so $l = k$ since $d \trianglelefteq a_k$. It follows that $d \trianglelefteq a_k$ where $a_k = a_l \leq a_i$. Thus d occurs in (5) and we have finished the proof. $\square$

We should note that Breslin and Ligh [3] have derived a formula for $(S)_f$ for any subset $S \subset \mathbb{P}$ in terms of an arbitrary lower-closed set containing S. Since this identity expresses $(S)_f$ as a product of two (not necessarily square) matrices, one can appy the Cauchy-Binet formula to it, as done in Li's paper [20], to obtain $\det(S)_f$ as a sum of determinants. This approach was also generalized to meet semi-lattices in an article of Haukkanen [14].

5. Comments and an open question

Our results can also yield other information about the matrices $(P)_{FG}$ and $(L)_f$. The following theorem is an example of this. In it, we specialize the ring to be the complex numbers $\mathbb{C}$ for simplicity, although more general rings can be used.

Theorem 5.1. *Let P be a finite poset and let $F, G \in I(P, \mathbb{C})$.*

1. *The matrix $(P)_{FG}$ is invertible if and only if $F(a,a), G(a,a) \neq 0$ for all $a \in P$.*
2. *The matrix $(P)_{FG}$ is positive definite if and only if $F(a,a)G(a,a) > 0$ for all $a \in P$.*

Proof. Part (1) follows immediately from Theorem 2.2 and the fact that a matrix over $\mathbb{C}$ is invertible if and only if its determinant is nonzero.

For part (2), let the total order used to index the rows and columns of P be a linear extension of P. Then each principal submatrix of P is indexed by a lower-closed subset S of P. It follows that the submatrix indexed by S is exactly $(S)_{FG}$. But now we are done by Theorem 2.2 again, since a matrix is positive definite if and only if the principal subdeterminants are all positive. $\square$

If L is a finite lattice, then one can compute the determinant of a matrix involving joins (least upper bounds) $a \vee b$ of elements $a, b \in L$ by working in the dual of L so that joins become meets. For general results about join and meet matrices, see the paper of Korkee and Haukkanen [18].

However, there are factorizations of such determinants which we have been unable to obtain by our methods. As an example, we consider the matrix $T_n(q)$ of chromatic joins introduced by Birkhoff and Lewis [5] to study the chromatic polynomial of planar maps. To define this matrix, let q be a formal parameter. Let Π_n denote the lattice of partitions of $\{1, 2, \ldots, n\}$ ordered by refinement and let NC_n denote the lattice of noncrossing partitions of the same set. (An excellent survey about noncrossing partitions can be found in the article of Simion [24].) Then Tutte [31] showed that the matrix of chromatic joins could be defined as

$$T_n(q) = \left(q^{\mathrm{bk}(a \vee_{\Pi_n} b)}\right)_{a,b \in NC_n}$$

where $\mathrm{bk}(a)$ is the number of blocks (subsets) in the partition a.

Tutte [30] also derived a product formula for the determinant of $T_n(q)$ in terms of Beraha polynomials. (See also Dahab [9].) Letting $\lfloor \cdot \rfloor$ denote the floor or round down function, the *nth Beraha polynomial* is defined to be $p_0(q) = 0$ for $n = 0$, and for $n \geq 1$

$$p_n(q) = \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i-1}{i} q^{\lfloor n/2 \rfloor - i}.$$

Using the version of Tutte's formula in the paper of Copeland, Schmidt, and Simion [8] gives

$$\det T_n(q) = q^{\binom{2n-1}{n}} \prod_{m=1}^{n-1} \left[\frac{p_{m+2}(q)}{q p_m(q)} \right]^{\frac{m+1}{n} \binom{2n}{n-m-1}}.$$

The same paper also contains a related determinant-product identity which the authors note that they were unable to prove using Lindström's Theorem (Theorem 3.1 above), although other proofs exist. It would be interesting to find a way to apply the machinery of this paper to these identities.

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References

[1] T. M. Apostol, Arithmetical properties of generalized Ramanujan sums, *Pacific J. Math.* **41** (1972), 281–293.

[2] S. Beslin and S. Ligh, Another generalization of Smith's determinant, *Bull. Austral. Math. Soc.* **40** (1989), 413–415.

[3] S. Beslin and S. Ligh, Greatest common divisor matrices, *Linear Algebra Appl.* **118** (1989), 69–76.

[4] B. V. Bhat, On greatest common divisor matrices and their applications, *Linear Algebra Appl.* **158** (1991), 77–97.

[5] G. D. Birkhoff and D. C. Lewis, Chromatic polynomials, *Trans. Amer. Math. Soc.* **60** (1946), 355–451.

[6] G. Chartrand and L. Lesniak, "Graphs and Digraphs," second edition, Wadsworth & Brooks/Cole, Pacific Grove, CA, 1986.

[7] G. F. Clements and B.Lindström, A sequence of (± 1)-determinants with large values, *Proc. Amer. Math. Soc.* **16** (1965) 548–550.

[8] A. Copeland, F. Schmidt, and R. Simion, Note on two determinants with interesting factorizations, *Discrete Math.* **256** (2002), 449–458.

[9] R. Dahab, "The Birkhoff-Lewis Equations," Ph.D. thesis, University of Waterloo, Waterloo, ONT, 1993.

[10] G. Daniloff, Contribution à la théorie des fonctions arithmétiques (in Bulgarian; French summary), *Sb. Bulgar. Akad. Nauk* **35** (1942), 479–590.

[11] I. Gessel and G. Viennot, Binomial determinants, paths, and hook length formulae, *Adv. in Math.* **58** (1985), 300–321.

[12] I. Gessel and G. Viennot, Determinants, paths, and plane partitions, preprint, available at **http://www.cs.brandeis.edu/~ira**.

[13] F. Harary, "Graph Theory," Addison-Wesley, Reading, MA, 1971.

[14] P. Haukkanen, On meet matrices on posets, *Linear Algebra Appl.* **249** (1996), 111–123.

[15] P. Haukkanen, J. Wang, and J. Sillanpää, On Smith's determinant, *Linear Algebra Appl.* **258** (1997), 251–269.

[16] H. Jager, The unitary analogues of some identities for certain arithmetical functions, *Neder. Akad. Wetensch. Proc. Ser. A* **64** (1961), 508–515.

[17] S. Karlin and J. L. McGregor, Coincidence probabilities, *Pacific J. Math.* **9** (1959), 1141–1164.

[18] I. Korkee and P. Haukkanen, On meet matrices and join matrices associated with incidence functions, *Linear Algebra Appl.* **372** (2003), 127–153.

[19] C, Krattenthaler, Watermelon configurations with wall interaction: exact and asymptotic results, preprint, available at **http://igd.univ-lyon1.fr/˜webwuler/home/kratt/papers.html**.

[20] Z. Li, The determinants of GCD matrices, *Linear Algebra Appl.* **134** (1990), 137–143.

[21] B. Lindström, Determinants on semilattices, *Proc. Amer. Math. Soc.* **20** (1969), 207–208.

[22] B. Lindström, On the vector representation of induced matroids, *Bull. London Math. Soc.* **5** (1973), 85–90.

[23] G.-C. Rota, On the foundations of combinatorial theory I. Theory of Möbius functions, *Z. Wahrscheinlichkeitstheorie* **2** (1964), 340–368.

[24] R. Simion, Noncrossing partitions, *Discrete Math.* **217** (2000), 367–409.

[25] J. C. Slater, The theory of complex spectra, *Phys. Rev.* **34** (1929), 1293–1322.

[26] D. A. Smith, Bivariate function algebras on posets, *J. Reine Angew. Math.* **251** (1971) , 100–109.

[27] H. J. S. Smith, On the value of a certain arithmetical determinant, *Proc. London Math. Soc. Ser. 1* **7** (1876), 208–212.

[28] R. P. Stanley, "Enumerative Combinatorics, Volume 1," Cambridge University Press, Cambridge, 1997.

[29] J. R. Stembridge, Nonintersecting paths, pfaffians and plane partitions, *Adv. in Math.* **83** (1990), 96–131.

[30] W. T. Tutte, The matrix of chromatic joins, *J. Combin. Theory Ser. B* **57** (1993), 269–288.

[31] W. T. Tutte, On the Birkhoff-Lewis equation, *Discrete Math.* **92** (1991), 417–425.

[32] H. S. Wilf, Hadamard determinants, möbius functions, and the chromatic number of a graph, *Bull. Amer. Math. Soc.* **74** (1968), 960–964.

Department of Mathematics, Faculty of Arts and Sciences, Gazi University, 06500 Teknikokullar - Ankara, TURKEY

E-mail address: `ealtinisik@gazi.edu.tr`

Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA

E-mail address: `sagan@math.msu.edu`

Department of Mathematics, Faculty of Arts and Sciences, Gazi University, 06500 Teknikokullar - Ankara, TURKEY

E-mail address: `naimtuglu@gazi.edu.tr`