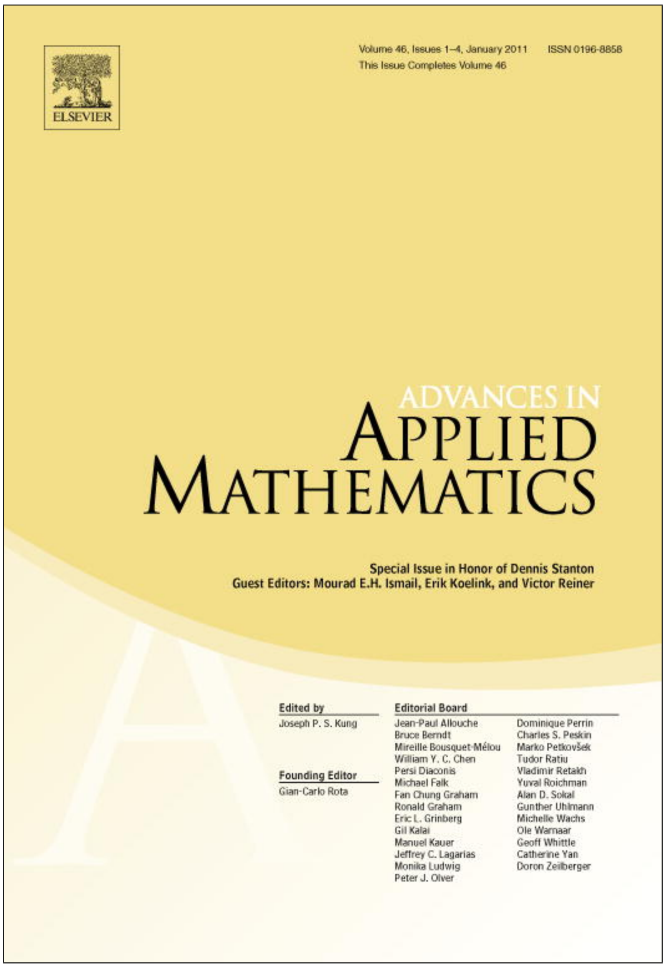
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Eulerian quasisymmetric functions and cyclic sieving

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ABSTRACT

It is shown that a refined version of a q-analogue of the Eulerian numbers together with the action, by conjugation, of the subgroup of the symmetric group S_n generated by the n-cycle $(1, 2, \ldots, n)$ on the set of permutations of fixed cycle type and fixed number of excedances provides an instance of the cyclic sieving phenomenon of Reiner, Stanton and White. The main tool is a class of symmetric functions recently introduced in work of two of the authors.

1. Introduction

In [15,16], certain quasisymmetric functions, called "Eulerian quasisymmetric functions" are introduced and shown to be in fact symmetric functions. These symmetric functions have been useful in the study of the joint distribution of two permutation statistics, major index and excedance number. There are various versions of the Eulerian quasisymmetric functions. They are defined by first associating a fundamental quasisymmetric function with each permutation in the symmetric group S_n and then summing these fundamental quasisymmetric functions over permutations in certain subsets

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of S_n. To obtain the most refined versions, the cycle-type Eulerian quasisymmetric functions $Q_{\lambda,j}$, one sums the fundamental quasisymmetric functions associated with the permutations having exactly j excedances and cycle type λ. By summing over all the permutations in S_n having j excedances and k fixed points, one obtains the less refined version $Q_{n,j,k}$. The precise definition of the Eulerian quasisymmetric functions and other key terms can be found in Section 2.

Shareshian and Wachs [15,16] derive a formula for the generating function of $Q_{n,j,k}$ which specializes to a (q,r)-analog of a classical formula for the exponential generating function of the Eulerian polynomials. The (q,r)-analogue of the classical formula is given by

$$1+\sum_{n\geqslant 1} A_n^{\mathrm{maj,exc,fix}}(q,t,r)\frac{z^n}{[n]_q!}=\frac{(1-tq)\exp_q(rz)}{\exp_q(ztq)-tq\exp_q(z)}, \tag{1}$$

where

$$A_n^{\mathrm{maj,exc,fix}}(q,t,r):=\sum_{\sigma\in S_n} q^{\mathrm{maj}(\sigma)}t^{\mathrm{exc}(\sigma)}r^{\mathrm{fix}(\sigma)},$$

and

$$\exp_q(z):=\sum_{n\geqslant 0}\frac{z^n}{[n]_q!}.$$

The cycle-type Eulerian quasisymmetric functions $Q_{\lambda,j}$ remain somewhat mysterious, and one might expect that better understanding of them will lead to further results on permutation statistics. In this paper, we provide evidence that this expectation is reasonable. With Theorem 4.1, we prove a conjecture from [16], describing the expansion of $Q_{\lambda,j}$, for $\lambda=(n)$, in terms of the power sum basis for the space of symmetric functions. Combining Theorem 4.1 with a technique of Désarménien [4], we are able to evaluate, at all nth roots of unity, the *cycle-type q-Eulerian numbers*

$$a_{\lambda,j}(q):=\sum_{\sigma\in S_{\lambda,j}} q^{\mathrm{maj}(\sigma)-\mathrm{exc}(\sigma)}, \tag{2}$$

where $S_{\lambda,j}$ is the set of all $\sigma\in S_n$ having exactly j excedances and cycle type λ. This and an analysis of the excedance statistic on the centralizers $C_{S_n}(\tau)$ of certain permutations $\tau\in S_n$ enable us to establish the relationship between the polynomials $a_{\lambda,j}(q)$ and the cyclic sieving phenomenon of Reiner, Stanton and White [11] given in Theorem 1.2 below.

Notation 1.1. For a positive integer d, ω_d will denote throughout this paper an arbitrary complex primitive dth root of 1.

Theorem 1.2. *Let $\gamma_n=(1,2,\dots,n)\in S_n$ and let $G_n=\langle\gamma_n\rangle\leqslant S_n$. Then for all partitions λ of n and $j\in\{0,1,\dots,n-1\}$, the group G_n acts on $S_{\lambda,j}$ by conjugation and the triple $(G_n,S_{\lambda,j},a_{\lambda,j}(q))$ exhibits the cyclic sieving phenomenon. In other words, if $\tau\in G_n$ has order d then*

$$a_{\lambda,j}(\omega_d)=\left|C_{S_n}(\tau)\cap S_{\lambda,j}\right|. \tag{3}$$

For λ a partition of n, let S_λ be the set of all $\sigma\in S_n$ of cycle type λ and define the *cycle type Eulerian polynomial* associated with λ as

$$A_\lambda^{\mathrm{maj,exc}}(q,t):=\sum_{\sigma\in S_\lambda} q^{\mathrm{maj}(\sigma)}t^{\mathrm{exc}(\sigma)}.$$

Then (3) can be rewritten as

$$A_{\lambda}^{\mathrm{maj,exc}}\big(\omega_d, t\omega_d^{-1}\big) = \sum_{\sigma \in C_{S_n}(\tau) \cap S_\lambda} t^{\mathrm{exc}(\sigma)}, \tag{4}$$

which is clearly a refinement of

$$A_n^{\mathrm{maj,exc,fix}}\big(\omega_d, t\omega_d^{-1}, s\big) = \sum_{\sigma \in C_{S_n}(\tau)} t^{\mathrm{exc}(\sigma)} s^{\mathrm{fix}(\sigma)}. \tag{5}$$

Theorem 6.1 below says that both sides of (5) are, in fact, equal to

$$A_{\frac{n}{d}}^{\mathrm{maj,exc,fix}}\left(1, t, \frac{s^d + t[d-1]_t}{[d]_t}\right) [d]_t^{\frac{n}{d}}, \tag{6}$$

which by setting $s = 1$ yields

$$A_n^{\mathrm{maj,exc}}\big(\omega_d, t\omega_d^{-1}\big) = A_{\frac{n}{d}}(t)\, [d]_t^{\frac{n}{d}},$$

for all divisors d of n. For cycle-type Eulerian polynomials and all divisors d of n, we obtain in Theorem 6.3 the similar looking results,

$$A_{(n)}^{\mathrm{maj,exc}}\big(\omega_d, t\omega_d^{-1}\big) = \big(tA_{\frac{n}{d}-1}(t)\, [d]_t^{\frac{n}{d}}\big)_d, \tag{7}$$

and

$$A_{(n+1)}^{\mathrm{maj,exc}}\big(\omega_d, t\omega_d^{-1}\big) = tA_{\frac{n}{d}}(t)\, [d]_t^{\frac{n}{d}}, \tag{8}$$

where $(f(t))_d$ is defined in (18) for each polynomial $f(t)$ and positive integer d.

The paper is organized as follows. In Section 2, we review definitions of various terms such as cyclic sieving and Eulerian quasisymmetric functions. We also present some preliminary results on Eulerian quasisymmetric functions from [16]. In Section 3 we describe a technique that uses symmetric function theory to evaluate certain polynomials at roots of unity based on work of Désarménien [4]. Theorem 4.1 mentioned above is proved in Section 4 by means of results of [16] which enable one to express the cycle-type Eulerian quasisymmetric functions in terms of the less refined version of Eulerian quasisymmetric functions. The proof of Theorem 1.2 appears in Section 5. In Section 6, we prove that both sides of (5) are equal to (6), that (7) and (8) hold, and that another triple exhibits the cyclic sieving phenomenon, namely $(G_n, S_{n,j}, a_{(n+1),j+1}(q))$, where $S_{n,j}$ is the set of all permutations in S_n with j excedances.

2. Definitions, known facts and preliminary results

2.1. Cyclic sieving

Let G be a finite cyclic group acting on a set X, and let $f(q)$ be a polynomial in q with nonnegative integer coefficients. For $g \in G$, let $\mathrm{Fix}(g)$ be the set of fixed points of g in X. The triple $(G, X, f(q))$ *exhibits the cyclic sieving phenomenon* of Reiner, Stanton and White [11] if for each $g \in G$ we have

$$f(\omega_{|g|}) = \big|\mathrm{Fix}(g)\big|, \tag{9}$$

where $|g|$ is the order of g.

Remark 2.1. Since all elements of order d in a cyclic group G generate the same subgroup, they have the same set of fixed points in any action. Thus our formulation of the cyclic sieving phenomenon is equivalent to the definition given in [11].

Note that if $(G, X, f(q))$ exhibits the cyclic sieving phenomenon then $f(1) = |X|$, and interesting examples arise where $f(q)$ is the generating function for some natural statistic on X, that is, there exists some useful function $s : X \to \mathbb{N}$ such that

$$f(q) = \sum_{x \in X} q^{s(x)}.$$

See [11] and [6] for many examples. Recent work on this subject appears in [10,5,1,2,12,8,9].

2.2. Permutation statistics

Recall that for a permutation $\sigma \in S_n$ acting from the right on $[n] := \{1, \dots, n\}$, the *excedance set* of σ is

$$\mathrm{Exc}(\sigma) := \{i \in [n-1]\colon\ i\sigma > i\}$$

and the *descent set* of σ is

$$\mathrm{Des}(\sigma) := \{i \in [n-1]\colon\ i\sigma > (i+1)\sigma\}.$$

The *major index* of σ is

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{Des}(\sigma)} i,$$

and the *excedance and descent statistics* of σ are, respectively,

$$\mathrm{exc}(\sigma) := \left|\mathrm{Exc}(\sigma)\right|,$$

and

$$\mathrm{des}(\sigma) := \left|\mathrm{Des}(\sigma)\right|.$$

Let $\mathrm{Fix}(\sigma)$ denote the set of fixed points of σ, that is

$$\mathrm{Fix}(\sigma) := \{i \in [n]\colon\ i\sigma = i\}$$

and let

$$\mathrm{fix}(\sigma) := \left|\mathrm{Fix}(\sigma)\right|.$$

The excedance and descent statistics are equidistributed, and for positive integer n, the nth *Eulerian polynomial* $A_n(t)$ can be defined as

$$\sum_{\sigma \in S_n} t^{\mathrm{exc}(\sigma)} = A_n(t) = \sum_{\sigma \in S_n} t^{\mathrm{des}(\sigma)}.$$

For convenience we set

$$A_0(t) := t^{-1}.$$

The Eulerian polynomial is also the generating polynomial for the *ascent statistic* on S_n, $\operatorname{asc}(\sigma) := |\{i \in [n-1]\colon\ i\sigma < (i+1)\sigma\}|$.

For permutation statistics $s_1, \ldots, s_k$ and a positive integer n, define the polynomial

$$A_n^{s_1,\ldots,s_k}(t_1,\ldots,t_k) := \sum_{\sigma \in S_n} t_1^{s_1(\sigma)} t_2^{s_2(\sigma)} \cdots t_k^{s_k(\sigma)}.$$

2.3. Partitions and symmetric functions

We use standard notation for partitions and symmetric functions. References for basic facts are [7, 18,13]. In particular, p_λ and h_λ will denote, respectively, the power sum and complete homogeneous symmetric functions associated to a partition λ. We use $l(\lambda)$ to denote the number of (nonzero) parts of λ and $m_j(\lambda)$ to denote the number of parts of λ equal to j. We write $\operatorname{Par}(n)$ for the set of all partitions of n. For $\lambda \in \operatorname{Par}(n)$, define the number

$$z_\lambda := \prod_{j=1}^{n} j^{m_j(\lambda)} m_j(\lambda)!.$$

We use two standard methods to describe a partition $\lambda \in \operatorname{Par}(n)$. The first is to write $\lambda = (\lambda_1, \ldots, \lambda_{l(\lambda)})$, listing the (nonzero) parts of λ so that $\lambda_i \geqslant \lambda_{i+1}$ for all i. The second is to write $\lambda = 1^{m_1(\lambda)} \ldots n^{m_n(\lambda)}$, usually suppressing those symbols $i^{m_i(\lambda)}$ such that $m_i(\lambda) = 0$ and writing i^1 as simply i. In particular, if $n = dk$ then d^k represents the partition with k parts of size d and no other parts. If $\lambda = (\lambda_1, \ldots, \lambda_k) \in \operatorname{Par}(n)$ and $q \in \mathbb{Q}$ with $q\lambda_i \in \mathbb{P}$ for all $i \in [k]$, we write $q\lambda$ for $(q\lambda_1, \ldots, q\lambda_k) \in \operatorname{Par}(qn)$.

For each $\sigma \in S_n$, let $\lambda(\sigma)$ denote the cycle type of σ. Given $\lambda \in \operatorname{Par}(n)$, we write S_λ for the set of all $\sigma \in S_n$ having cycle type λ. As in (2), we write $S_{\lambda,j}$ for the set of those $\sigma \in S_\lambda$ satisfying $\operatorname{exc}(\sigma) = j$.

For symmetric functions f, g with coefficients in $\mathbb{Q}[t]$, $f[g]$ will denote the plethysm of g by f. The same notation will be used for plethysm of symmetric power series with no bound on their degree. One such power series is $\mathsf{H} := \sum_{n \geqslant 1} \mathsf{h}_n$. If we set

$$\begin{aligned}\mathsf{L} &:= \sum_{d \geqslant 1} \frac{\mu(d)}{d} \log(1 + \mathsf{p}_d) \\ &= \sum_{d \geqslant 1} \frac{\mu(d)}{d} \sum_{i \geqslant 1} \frac{(-1)^{i-1}}{i} \mathsf{p}_d^i,\end{aligned}$$

where μ is the classical Möbius function, then H and L are plethystic inverses, that is,

$$\mathsf{L}\big[\mathsf{H}[f]\big] = \mathsf{H}\big[\mathsf{L}[f]\big] = f \tag{10}$$

for all symmetric power series f. (This is due to Cadogan, see [3] or [18, Exercise 7.88e].) Note also that for any power series $h(t, x_1, x_2, \ldots)$ with coefficients in $\mathbb{Q}$ that is symmetric in $x_1, x_2, \ldots$ and any $d \in \mathbb{P}$, we have

$$\mathsf{p}_d[h] = h\big(t^d, x_1^d, x_2^d, \ldots\big). \tag{11}$$

We shall use without further mention the facts $(f + g)[h] = f[h] + g[h]$ and $(fg)[h] = f[h]g[h]$.

2.4. q-Analogues

We use the standard notation for polynomial analogues of positive integers, that is, for a positive integer n and a variable q, we define

$$[n]_q := \sum_{j=0}^{n-1} q^j = \frac{1-q^n}{1-q}$$

and

$$[n]_q! := [n]_q[n-1]_q \cdots [1]_q.$$

Also define

$$[0]_q! := 1.$$

It is well known that for any sequence $(k_1, \dots, k_m)$ of nonnegative integers whose sum is n, the q-multinomial coefficient

$$\begin{bmatrix} n \\ k_1, \dots, k_m \end{bmatrix}_q := \frac{[n]_q!}{[k_1]_q! \cdots [k_m]_q!}$$

is always a polynomial in $\mathbb{N}[q]$. The following q-analogue of the multinomial version of the Pascal recurrence relation is also well known (see [17, (17b)]):

$$\begin{bmatrix} n \\ k_1, \dots, k_m \end{bmatrix}_q = \sum_{i=1}^{m} q^{k_{i+1}+\dots+k_m} \begin{bmatrix} n-1 \\ k_1, \dots, k_i - 1, \dots, k_m \end{bmatrix}_q, \tag{12}$$

where $(k_1, \dots, k_m)$ is a sequence of positive integers whose sum is n. We will need the following elementary fact, which also plays a role in the work of Reiner, Stanton and White [11].

Proposition 2.2. *(See [11, Eq. (4.5)].) Let $(k_1, \dots, k_m)$ be a sequence of nonnegative integers whose sum is n. If $d|n$ then*

$$\begin{bmatrix} n \\ k_1, \dots, k_m \end{bmatrix}_q \Bigg|_{q=\omega_d} = \begin{cases} \dbinom{\frac{n}{d}}{\frac{k_1}{d}, \dots, \frac{k_m}{d}} & \text{if } d|k_i \ \forall i \in [m], \\ 0 & \text{otherwise.} \end{cases}$$

2.5. The Eulerian quasisymmetric functions

Given a permutation $\sigma \in S_n$, we write σ in one line notation,

$$\sigma = \sigma_1 \dots \sigma_n,$$

where $\sigma_i = i\sigma$. Set

$$[\bar{n}] := \{\bar{\imath}\colon\ i \in [n]\},$$

and let $w(\sigma)$ be the word in the alphabet $\mathcal{A} := [n] \cup [\bar{n}]$ obtained from σ by replacing σ_i with $\overline{\sigma_i}$ whenever $i \in \mathrm{Exc}(\sigma)$. Order $\mathcal{A}$ by

$$\bar{1} < \cdots < \bar{n} < 1 < \cdots < n,$$

and for any word $w := w_1 \dots w_n$ from $\mathcal{A}$, set

$$\mathrm{Des}(w) := \bigl\{ i \in [n-1]\colon\ w_i > w_{i+1} \bigr\}.$$

Now, for $\sigma \in S_n$, define

$$\mathrm{Dex}(\sigma) := \mathrm{Des}\bigl(w(\sigma)\bigr).$$

For example, if $\sigma = 641532$ then $w(\sigma) = \bar{6}\bar{4}1\bar{5}32$ and $\mathrm{Dex}(\sigma) = \{1, 3, 5\}$.

Recall now that a *quasisymmetric function* is a power series (with rational coefficients) f of bounded degree in variables $x_1, x_2, \dots$ such that if $j_1 < \cdots < j_k$ and $l_1 < \cdots < l_k$, then for all $a_1, \dots, a_k$ the coefficients in f of $\prod_{i=1}^k x_{j_i}^{a_i}$ and $\prod_{i=1}^k x_{l_i}^{a_i}$ are equal. The usual addition, multiplication and scalar multiplication make the set $\mathcal{Q}$ of quasisymmetric functions a $\mathbb{Q}$-algebra that strictly contains the algebra of symmetric functions. For $n \in \mathbb{P}$ and $S \subseteq [n-1]$, set

$$\mathrm{Mon}_S := \left\{ \prod_{i=1}^n x_{j_i}\colon\ j_i \geqslant j_{i+1} \text{ for } i \in [n-1] \text{ and } j_i > j_{i+1} \text{ if } i \in S \right\},$$

and define the *fundamental quasisymmetric function* associated with S to be

$$F_S := \sum_{x \in \mathrm{Mon}_S} x \in \mathcal{Q}.$$

Recall from above that we have defined $S_{\lambda,j}$ to be the set of all permutations of cycle type λ with j excedances. The *Eulerian quasisymmetric function* associated to the pair (λ, j) is

$$Q_{\lambda,j} := \sum_{\sigma \in S_{\lambda,j}} F_{\mathrm{Dex}(\sigma)} \in \mathcal{Q}.$$

The Eulerian quasisymmetric functions were introduced in [16] as a tool for studying the (maj, exc) q-analogue of the Eulerian polynomials. The connection between the Eulerian quasisymmetric functions and the q-Eulerian numbers is given in the following proposition. The *stable principal specialization* ps is a homomorphism from the algebra of quasisymmetric functions $\mathcal{Q}$ to the algebra of formal power series $\mathbb{Q}[[q]]$ defined by $\mathrm{ps}(x_i) = q^{i-1}$.

Proposition 2.3. *(See [16, Eq. (2.13)].)*[3] *For all partitions λ of n and $j \in \{0, 1, \dots, n-1\}$, let $a_{\lambda,j}(q)$ be as in* (2)*. Then*

$$\mathrm{ps}(Q_{\lambda,j}) = \frac{a_{\lambda,j}(q)}{\prod_{i=1}^n (1-q^i)}.$$

[3] Eq. (2.13) in [16] has an extra factor of q^j because $a_{\lambda,j}(q)$ is defined there to be the maj enumerator of $S_{\lambda,j}$ rather than the maj–exc enumerator.

In [16], it is also shown that in fact $Q_{\lambda,j}$ is always a symmetric function. If one knows $Q_{(n),j}$ for all n, j, then a fairly compact explicit formula for each $Q_{\lambda,j}$ can be obtained from Corollary 6.1 of [16], which says that for any $\lambda \in \mathrm{Par}(n)$,

$$\sum_{j=0}^{n-1} Q_{\lambda,j} t^j = \prod_{i=1}^{n} \mathsf{h}_{m_i(\lambda)}\left[\sum_{l=0}^{i-1} Q_{(i),l} t^l\right]. \tag{13}$$

As noted in [16], if we set

$$Q_{n,j} := \sum_{\lambda \in \mathrm{Par}(n)} Q_{\lambda,j}$$

for $n \geqslant 1$, and

$$Q_{0,0} = Q_{(0),0} = 1,$$

Eq. (13) implies

$$\sum_{n,j \geqslant 0} Q_{n,j} t^j = \sum_{n \geqslant 0} \mathsf{h}_n\left[\sum_{i,j \geqslant 0} Q_{(i),j} t^j\right] = \mathsf{H}\left[\sum_{i,j \geqslant 0} Q_{(i),j} t^j\right],$$

which by (10) is equivalent to

$$\sum_{n,j \geqslant 0} Q_{(n),j} t^j = \mathsf{L}\left[\sum_{i,j \geqslant 0} Q_{i,j} t^j\right]. \tag{14}$$

Proposition 6.6 of [16] gives an explicit formula for $Q_{n,j}$ in terms of the power sum symmetric function basis,

$$\sum_{j=0}^{n-1} Q_{n,j} t^j = \sum_{\nu \in \mathrm{Par}(n)} z_\nu^{-1} A_{l(\nu)}(t) \prod_{i=1}^{l(\nu)} [\nu_i]_t \mathsf{p}_\nu. \tag{15}$$

By combining (13), (14) and (15) we obtain a formula for each $Q_{\lambda,j}$, which will be used in Section 4 to prove a conjecture from [16] giving the expansion of $Q_{(n),j}$ in the power sum basis.

3. A symmetric function technique

We describe here a general technique for evaluating polynomials at roots of unity based on a technique of Désarménien [4]. This technique provides a key step in our proof of Theorem 1.2. One can also prove Theorem 1.1 using Springer's theory of regular elements in place of the technique we give here. A description of the relevance of Springer's work to the cyclic sieving phenomenon appears in [11].

Given a homogeneous symmetric function F of degree n and a partition ν of n, let χ_ν^F be the coefficient of $z_\nu^{-1}\mathsf{p}_\nu$ in the expansion of F in terms of the basis $\{z_\nu^{-1}\mathsf{p}_\nu\colon \nu \in \mathrm{Par}(n)\}$ for the space of homogeneous symmetric functions of degree n. That is, χ_ν^F is uniquely determined by

$$F = \sum_{\nu \in \mathrm{Par}(n)} \chi_\nu^F z_\nu^{-1} \mathsf{p}_\nu.$$

Although we will not make use of this, we note that if F is the Frobenius characteristic of a class function of S_n then χ_ν^F is the value of the class function on permutations of cycle type ν. Recall that ps denotes the stable principal specialization defined in Section 2.5.

The following result is implicit in [4].

Proposition 3.1. *Suppose $f(q) \in \mathbb{Q}[q]$ and there exists a homogeneous symmetric function F of degree n with coefficients in $\mathbb{Q}$ such that*

$$f(q) = \prod_{i=1}^{n} \left(1 - q^i\right) \operatorname{ps}(F).$$

Then for all $d, k \in \mathbb{P}$ such that $n \in \{dk, dk+1\}$,

$$f(\omega_d) = \chi_\nu^F,$$

where $\nu = d^k$ or $\nu = 1d^k$.

Proof. By expanding F in the power sum basis for the symmetric functions, we have,

$$\begin{aligned} f(q) &= \prod_{i=1}^{n} \left(1 - q^i\right) \sum_{\mu \in \operatorname{Par}(n)} \chi_\mu^F z_\mu^{-1} \operatorname{ps}(\mathrm{p}_\mu) \\ &= \sum_{\mu \in \operatorname{Par}(n)} \chi_\mu^F z_\mu^{-1} \frac{\prod_{i=1}^{n}(1-q^i)}{\prod_{i=1}^{l(\mu)}(1-q^{\mu_i})}. \end{aligned} \tag{16}$$

It is shown in [4, Proposition 7.2] that for all $\mu \in \operatorname{Par}(n)$,

$$T_\mu(q) := \frac{\prod_{i=1}^{n}(1-q^i)}{\prod_{i=1}^{l(\mu)}(1-q^{\mu_i})}$$

is a polynomial in q whose value at ω_d is given by

$$T_\mu(\omega_d) = \begin{cases} z_\mu & \text{if } \mu = d^k \text{ or } \mu = 1d^k, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

We include a proof for the sake of completeness. Since

$$T_\mu(q) = \begin{bmatrix} n \\ \mu_1, \ldots, \mu_{l(\mu)} \end{bmatrix}_q \prod_{i=1}^{l(\mu)} \prod_{j=1}^{\mu_i - 1} \left(1 - q^j\right),$$

we see that $T_\mu(q)$ is a polynomial and that if $T_\mu(\omega_d) \neq 0$ then $\mu_i \leqslant d$ for all i. Hence, in the case that $n = dk$, it follows from Proposition 2.2 that $T_\mu(\omega_d) \neq 0$ only if $\mu_i = d$ for all i. By Proposition 2.2,

$$T_{d^k}(\omega_d) = k! \left(\prod_{j=1}^{d-1} \left(1 - \omega_d^j\right) \right)^k = k! d^k.$$

Similarly, in the case that $n = dk + 1$, we use (12) and Proposition 2.2 to show that $T_\mu(\omega_d)$ equals $k!d^k$ if $\mu = 1d^k$ and is 0 otherwise. Hence, in either case, (17) holds. Now by plugging (17) into (16) we obtain the desired result. □

We will use Propostion 3.1 to evaluate the cycle-type Eulerian numbers $a_{\lambda,j}(q)$ at all the mth roots of unity, where $m \in \{n-1, n\}$. We see from Proposition 2.3 that we already have the required symmetric function, namely $Q_{\lambda,j}$. We thus obtain the first step in our proof of Theorem 1.2.

Proposition 3.2. *Let $\lambda \in \mathrm{Par}(n)$ and let $d, k \in \mathbb{P}$. If $dk = n$ then*

$$a_{\lambda,j}(\omega_d) = \chi_{d^k}^{Q_{\lambda,j}},$$

and if $dk = n - 1$ then

$$a_{\lambda,j}(\omega_d) = \chi_{1d^k}^{Q_{\lambda,j}}.$$

In [16] a formula for the coefficients $\chi_\nu^{Q_{(n),j}}$ is conjectured. This formula turns out to be just what we need to prove Theorem 1.2. In the next section we present the conjecture and its proof.

Remark 3.3. In [16] it is conjectured that $Q_{\lambda,j}$ is the Frobenius characteristic of some representation of S_n. A proof of this conjecture was recently obtained in [14]. By Proposition 3.2 and Theorem 1.2, the restriction of the representation in question to G_n is isomorphic to the permutation representation for the action of G_n on $S_{\lambda,j}$.

4. The expansion of $Q_{(n),j}$

In this section we present a key result of our paper (Theorem 4.1), which was conjectured in [16]. For a power series $f(t) = \sum_{j \geqslant 0} a_j t^j$ and an integer k, let $f(t)_k$ be the power series obtained from $f(t)$ by erasing all terms $a_j t^j$ such that $\gcd(j,k) \neq 1$, so

$$f(t)_k := \sum_{\gcd(j,k)=1} a_j t^j. \tag{18}$$

For example, if $f(t) = t + 3t^2 - 5t^3 + 7t^4$ then $f(t)_2 = t - 5t^3$.

For a partition $\nu = (\nu_1, \dots, \nu_k)$, set

$$g(\nu) := \gcd(\nu_1, \dots, \nu_k).$$

Theorem 4.1. *(See [16], Conjecture 6.5.) For $\nu = (\nu_1, \dots, \nu_k) \in \mathrm{Par}(n)$, set*

$$G_\nu(t) := \left(t A_{k-1}(t) \prod_{i=1}^{k} [\nu_i]_t \right)_{g(\nu)}.$$

Then

$$\sum_{j=0}^{n-1} Q_{(n),j} t^j = \sum_{\nu \in \mathrm{Par}(n)} z_\nu^{-1} G_\nu(t) \mathrm{p}_\nu. \tag{19}$$

Theorem 4.1 can be restated as follows. Since $Q_{(n),j}$ is a homogeneous symmetric function of degree n, it can be expanded in the basis $\{z_\lambda^{-1}\mathrm{p}_\lambda\colon \lambda \in \mathrm{Par}(n)\}$. Thus, the theorem says that the expansion coefficient of $z_\nu^{-1}\mathrm{p}_\nu$ is 0 if $\gcd(j, g(\nu)) \neq 1$, while if $\gcd(j, g(\nu)) = 1$ then the expansion coefficient equals the coefficient of t^j in $tA_{l(\nu)-1}(t)\prod_{i=1}^{l(\nu)}[\nu_i]_t$.

In order to prove Theorem 4.1 we need two lemmas. As above, we write μ for the classical Möbius function on $\mathbb{P}$, and recall that

$$\sum_{d|n}\mu(d) = \begin{cases} 1, & n=1, \\ 0, & \text{otherwise}. \end{cases} \tag{20}$$

Lemma 4.2. *For a partition $\nu = (\nu_1, \dots, \nu_l)$, we have*

$$G_\nu(t) = \sum_{d|g(\nu)} \mu(d) d^{l-1} t^d A_{l-1}\big(t^d\big) \prod_{i=1}^{l} \left[\frac{\nu_i}{d}\right]_{t^d}. \tag{21}$$

Proof. It is known (and follows, for example, from [17, Theorem 4.5.14]) that for any positive integer k we have

$$\frac{tA_{k-1}(t)}{(1-t)^k} = \sum_{j \geqslant 0} j^{k-1} t^j. \tag{22}$$

It follows directly from the definition of $f(t)_d$ that for any power series g, h and any $d \in \mathbb{P}$ we have

$$\big(g(t)h\big(t^d\big)\big)_d = g(t)_d h\big(t^d\big). \tag{23}$$

We see now that

$$\begin{aligned} G_\nu(t) &= \left(tA_{l-1}(t)\prod_{i=1}^{l}\frac{1-t^{\nu_i}}{1-t}\right)_{g(\nu)} \\ &= \left(\frac{tA_{l-1}(t)}{(1-t)^l}\prod_{i=1}^{l}\big(1-t^{\nu_i}\big)\right)_{g(\nu)} \\ &= \left(\sum_{j\geqslant 0} j^{l-1}t^j\right)_{g(\nu)} \prod_{i=1}^{l}\big(1-t^{\nu_i}\big) \\ &= \sum_{j\colon \gcd(g(\nu),j)=1} j^{l-1}t^j \prod_{i=1}^{l}\big(1-t^{\nu_i}\big), \end{aligned}$$

the third equality above following from (22) and (23).

Now

$$\begin{aligned} \sum_{d|g(\nu)}\mu(d)\sum_{a\geqslant 0}(ad)^{l-1}t^{ad} &= \sum_{j\geqslant 0} j^{l-1}t^j \sum_{d|\gcd(j,g(\nu))}\mu(d) \\ &= \sum_{j\colon \gcd(j,g(\nu))=1} j^{l-1}t^j, \end{aligned}$$

the second equality following from (20). We see now that

$$G_\nu(t) = \left(\sum_{d|g(\nu)} \mu(d) \sum_{a\geqslant 0} (ad)^{l-1} t^{ad}\right) \prod_{i=1}^{l} \left(1-t^{\nu_i}\right)$$
$$= \left(\sum_{d|g(\nu)} \mu(d) d^{l-1} \frac{t^d A_{l-1}(t^d)}{(1-t^d)^l}\right) \prod_{i=1}^{l} \left(1-t^{\nu_i}\right)$$
$$= \sum_{d|g(\nu)} \mu(d) d^{l-1} t^d A_{l-1}\left(t^d\right) \prod_{i=1}^{l} \frac{1-t^{\nu_i}}{1-t^d},$$

the second equality following from (22). □

Lemma 4.3. *We have*

$$1 + \sum_{k\geqslant 1} A_k(t) \frac{z^k}{k!} = \exp\left(\sum_{l\geqslant 1} t A_{l-1}(t) \frac{z^l}{l!}\right). \tag{24}$$

Proof. We apply the exponential formula (see [18, Corollary 5.1.6]) to the Eulerian polynomials. For any permutation σ in S_n let $\pi(\sigma)$ be the partition of the set $[n]$ whose blocks are the supports of the cycles in the cycle decomposition of σ. Let Π_n be the set of all partitions of the set $[n]$. For any partition π in Π_n set

$$A_\pi(t) := \sum_{\substack{\sigma\in S_n \\ \pi(\sigma)=\pi}} t^{\operatorname{exc}(\sigma)}.$$

Then

$$A_n(t) = \sum_{\pi\in\Pi_n} A_\pi(t),$$

and

$$A_\pi(t) = \prod_{i=1}^{k} A_{\{B_i\}}(t) = \prod_{i=1}^{k} A_{(|B_i|)}(t),$$

where $\pi = \{B_1, \dots, B_k\}$. It therefore follows from the exponential formula that

$$1 + \sum_{k\geqslant 1} A_k(t) \frac{z^k}{k!} = \exp\left(\sum_{l\geqslant 1} A_{(l)}(t) \frac{z^l}{l!}\right).$$

To complete the proof we observe that

$$A_{(l)}(t) = t A_{l-1}(t). \tag{25}$$

Indeed, if $l = 1$ then both sides of the equation are equal to 1. For $l > 1$ and $\sigma \in S_{(l)}$, write σ in cycle notation $(x_1, x_2, \dots, x_l)$ with $x_l = l$. Now let $v(\sigma) = x_1 \dots x_{l-1}$, a permutation in S_{l-1} in one line notation. The excedance set of σ is the union of $\{x_{l-1}\}$ and $\{x_i\colon i \text{ is an ascent of } \nu(\sigma)\}$. Since ν is a bijection from $S_{(l)}$ to S_{l-1}, Eq. (25) holds. $\square$

Proof of Theorem 4.1. We have

$$\begin{aligned}
\sum_{n\geqslant 1}\sum_{j=0}^{n-1} Q_{(n),j}t^j &= \mathsf{L}\Bigg[\sum_{i\geqslant 1}\sum_{j=0}^{i-1} Q_{i,j}t^j\Bigg]\\
&= \mathsf{L}\Bigg[\sum_{k\geqslant 1}\sum_{\nu\colon l(\nu)=k} z_\nu^{-1}A_k(t)\prod_{h=1}^{k}[\nu_h]_t \mathsf{p}_{\nu_h}\Bigg]\\
&= \sum_{d\geqslant 1}\frac{\mu(d)}{d}\sum_{i\geqslant 1}\frac{(-1)^{i-1}}{i}\mathsf{p}_d^i\Bigg[\sum_{k\geqslant 1}\sum_{\nu\colon l(\nu)=k} z_\nu^{-1}A_k(t)\prod_{h=1}^{k}[\nu_h]_t \mathsf{p}_{\nu_h}\Bigg]\\
&= \sum_{d\geqslant 1}\frac{\mu(d)}{d}\sum_{i\geqslant 1}\frac{(-1)^{i-1}}{i}\Bigg(\sum_{k\geqslant 1}\sum_{\nu\colon l(\nu)=k} z_\nu^{-1}A_k\big(t^d\big)\prod_{h=1}^{k}[\nu_h]_{t^d} \mathsf{p}_{d\nu_h}\Bigg)^i\\
&= \sum_{d\geqslant 1}\frac{\mu(d)}{d}\log\Bigg(1+\sum_{k\geqslant 1}\sum_{\nu\colon l(\nu)=k} z_\nu^{-1}A_k\big(t^d\big)\prod_{h=1}^{k}[\nu_h]_{t^d} \mathsf{p}_{d\nu_h}\Bigg),
\end{aligned} \tag{26}$$

the first equality following from (14), the second from (15), the third from the definition of L and the fourth from (11).

For any $k\in\mathbb{P}$, let $\mathcal{M}_k$ be the set of all sequences $a=(a_1,a_2,\ldots)$ of nonnegative integers such that $\sum_{r\geqslant 1}a_r=k$. Then

$$\begin{aligned}
\sum_{\nu\colon l(\nu)=k} z_\nu^{-1}\prod_{h=1}^{k}[\nu_h]_{t^d}\mathsf{p}_{d\nu_h} &= \sum_{\nu\colon l(\nu)=k}\frac{1}{\prod_{r\geqslant 1} m_r(\nu)!}\prod_{r\geqslant 1}\left(\frac{[r]_{t^d}\mathsf{p}_{dr}}{r}\right)^{m_r(\nu)}\\
&= \frac{1}{k!}\sum_{a\in\mathcal{M}_k}\binom{k}{a_1,a_2,\ldots}\prod_{r\geqslant 1}\left(\frac{[r]_{t^d}\mathsf{p}_{dr}}{r}\right)^{a_r}\\
&= \frac{1}{k!}\left(\sum_{r\geqslant 1}\frac{[r]_{t^d}\mathsf{p}_{dr}}{r}\right)^{k}.
\end{aligned} \tag{27}$$

We see now that

$$\begin{aligned}
\sum_{n\geqslant 1}\sum_{j=0}^{n-1} Q_{(n),j}t^j &= \sum_{d\geqslant 1}\frac{\mu(d)}{d}\log\left(1+\sum_{k\geqslant 1}\frac{A_k(t^d)}{k!}\left(\sum_{r\geqslant 1}\frac{[r]_{t^d}\mathsf{p}_{dr}}{r}\right)^k\right)\\
&= \sum_{d\geqslant 1}\frac{\mu(d)}{d}\sum_{k\geqslant 1}\frac{t^dA_{k-1}(t^d)}{k!}\left(\sum_{r\geqslant 1}\frac{[r]_{t^d}\mathsf{p}_{dr}}{r}\right)^k\\
&= \sum_{d\geqslant 1}\frac{\mu(d)}{d}\sum_{k\geqslant 1}\sum_{\nu\colon l(\nu)=k} z_\nu^{-1}t^dA_{k-1}\big(t^d\big)\prod_{i=1}^{k}[\nu_i]_{t^d}\mathsf{p}_{d\nu_i}
\end{aligned}$$

$$= \sum_{k\geqslant 1}\sum_{d\geqslant 1}\mu(d)d^{k-1}t^d A_{k-1}\big(t^d\big)\sum_{\nu:\, l(\nu)=k} z_{d\nu}^{-1}\prod_{i=1}^{k}[\nu_i]_{t^d}\mathsf{p}_{d\nu_i}$$

$$= \sum_{k\geqslant 1}\sum_{\nu:\, l(\nu)=k} z_\nu^{-1}\mathsf{p}_\nu \sum_{d|g(\nu)}\mu(d)d^{k-1}t^d A_{k-1}\big(t^d\big)\prod_{i=1}^{k}\left[\frac{\nu_i}{d}\right]_{t^d}$$

$$= \sum_{k\geqslant 1}\sum_{\nu:\, l(\nu)=k} z_\nu^{-1}\mathsf{p}_\nu G_\nu(t).$$

Indeed, first equality is obtained by combining (26) and (27), the second equality is obtained from (24), the third follows from (27), the fourth and fifth are obtained by straightforward manipulations, and the last follows from (21). □

5. The proof of Theorem 1.2

5.1. The expansion coefficients $\chi_{d^k}^{Q_{\lambda,j}}$

To compute the expansion coefficients $\chi_{d^k}^{Q_{\lambda,j}}$, we will need to obtain results like Theorem 4.1 with the partition (n) replaced by an arbitrary partition λ, but in such results we will only need the coefficients of power sum symmetric functions of the form $\mathsf{p}_{(d,\dots,d)}$. We begin with a definition generalizing that of $f(t)_d$. For a power series $f(t)=\sum_j a_j t^j$ and positive integers b,c, let $f(t)_{b,c}$ be the power series obtained from f by erasing all terms $a_i t^i$ such that $\gcd(i,b)\neq c$, so

$$f(t)_{b,c} := \sum_{\gcd(i,b)=c} a_i t^i.$$

For example, if $f(t)=1+2t+3t^2+4t^3+5t^4$ then $f(t)_{6,2}=3t^2+5t^4$. For any power series g,h, we have

$$\big(g(t)h\big(t^b\big)\big)_{b,c} = g(t)_{b,c}h\big(t^b\big). \tag{28}$$

We will use the following result.

Lemma 5.1. *Let $k,b,c\in\mathbb{P}$ and assume that $c|b$. Then*

$$\big(tA_{k-1}(t)[b]_t^k\big)_{b,c} = c^{k-1}\big(t^c A_{k-1}\big(t^c\big)[b/c]_{t^c}^k\big)_{b,c}. \tag{29}$$

Proof. We have

$$\begin{aligned}
\big(tA_{k-1}(t)[b]_t^k\big)_{b,c} &= \left(\frac{tA_{k-1}(t)}{(1-t)^k}\big(1-t^b\big)^k\right)_{b,c}\\
&= \left(\frac{tA_{k-1}(t)}{(1-t)^k}\right)_{b,c}\big(1-t^b\big)^k\\
&= \big(1-t^b\big)^k\sum_{j:\,\gcd(j,b)=c} j^{k-1}t^j\\
&= \big(1-t^b\big)^k\sum_{i:\,\gcd(i,b/c)=1} (ic)^{k-1}t^{ic}
\end{aligned}$$

$$\begin{aligned}
&= c^{k-1}\big(1-t^b\big)^k \sum_{i:\ \gcd(i,b/c)=1} i^{k-1} t^{ic} \\
&= c^{k-1}\big(1-t^b\big)^k \left(\frac{t^c A_{k-1}(t^c)}{(1-t^c)^k}\right)_{b,c} \\
&= c^{k-1}\left(t^c A_{k-1}\big(t^c\big)\frac{(1-t^b)^k}{(1-t^c)^k}\right)_{b,c} \\
&= c^{k-1}\big(t^c A_{k-1}\big(t^c\big)[b/c]^k_{t^c}\big)_{b,c}.
\end{aligned}$$

Indeed, the second and seventh equalities follow from (28), the third and sixth follow from (22), and the rest are straightforward. □

We begin our computation of $\chi^{Q_{\lambda,j}}_{d^k}$ by considering first the case where all parts of λ have the same size. For $\lambda, \nu \in \mathrm{Par}(n)$ and $j \in \{0, 1, \ldots, n-1\}$, set

$$\chi^{\lambda,j}_{\nu} := \chi^{Q_{\lambda,j}}_{\nu}$$

and

$$\pi_\nu := \binom{n}{\nu_1, \ldots, \nu_{l(\nu)}} \frac{1}{\prod_{j=1}^n m_j(\nu)!}. \tag{30}$$

Theorem 5.2. *Let $n, r, m, d, k \in \mathbb{P}$ with $n = rm = dk$. Set*

$$\mathrm{Par}(m; d, r) := \big\{\mu = (\mu_1, \ldots, \mu_{l(\mu)}) \in \mathrm{Par}(m)\colon\ \mu_i | d | r\mu_i \text{ for all } i \in \big[l(\mu)\big]\big\}.$$

Then

$$\sum_{j=0}^{n-1} \chi^{r^m,j}_{d^k} t^j = \sum_{\mu \in \mathrm{Par}(m;d,r)} \pi_{\frac{r}{d}\mu} \prod_{i=1}^{l(\mu)} \big(t A_{\frac{r}{d}\mu_i - 1}(t)[d]_t^{\frac{r}{d}\mu_i}\big)_{d,\mu_i}. \tag{31}$$

Proof. Note that (13) implies that

$$\sum_{j=0}^{n-1} Q_{r^m,j} t^j = \mathsf{h}_m\left[\sum_{j=0}^{r-1} Q_{(r),j} t^j\right]. \tag{32}$$

Now

$$\begin{aligned}
\mathsf{h}_m\left[\sum_{j=0}^{r-1} Q_{(r),j} t^j\right] &= \sum_{\mu \in \mathrm{Par}(m)} z_\mu^{-1} \mathsf{p}_\mu \left[\sum_{j=0}^{r-1} Q_{(r),j} t^j\right] \\
&= \sum_{\mu \in \mathrm{Par}(m)} z_\mu^{-1} \prod_{i=1}^{l(\mu)} \mathsf{p}_{\mu_i}\left[\sum_{\nu \in \mathrm{Par}(r)} z_\nu^{-1} G_\nu(t) \mathsf{p}_\nu\right] \\
&= \sum_{\mu \in \mathrm{Par}(m)} z_\mu^{-1} \prod_{i=1}^{l(\mu)} \left(\sum_{\nu \in \mathrm{Par}(r)} z_\nu^{-1} G_\nu\big(t^{\mu_i}\big) \mathsf{p}_{\mu_i \nu}\right).
\end{aligned}$$

Indeed, the first equality follows from the well-known expansion (see any of [7,13,18]) of h_m in the power sum basis, the second from Theorem 4.1 and the third from (11).

On the other hand, it follows from the definition of $\chi_\nu^{\lambda,j}$ that

$$\sum_{j=0}^{n-1} Q_{r^m,j} t^j = \sum_{\nu \in \mathrm{Par}(n)} z_\nu^{-1} \sum_{j \geqslant 0} \chi_\nu^{r^m,j} \mathsf{p}_\nu t^j.$$

We see now equating the coefficients of p_{d^k} on both sides of (32) yields

$$\sum_{j=0}^{n-1} \chi_{d^k}^{r^m,j} t^j = z_{d^k} \sum_{\mu \in \mathrm{Par}(m;d,r)} z_\mu^{-1} \prod_{i=1}^{l(\mu)} z^{-1}_{(\frac{d}{\mu_i})^{r\mu_i/d}} G_{(\frac{d}{\mu_i})^{r\mu_i/d}}\big(t^{\mu_i}\big). \tag{33}$$

Now for all $\mu \in \mathrm{Par}(m; d, r)$, we have

$$\begin{aligned} z_{d^k} z_\mu^{-1} \prod_{i=1}^{l(\mu)} z^{-1}_{(\frac{d}{\mu_i})^{r\mu_i/d}} &= \frac{d^k k!}{\prod_{j=1}^m m_j(\mu)! \prod_{i=1}^{l(\mu)} \mu_i (\frac{r\mu_i}{d})! (\frac{d}{\mu_i})^{r\mu_i/d}} \\ &= \frac{k! \prod_{i=1}^{l(\mu)} \mu_i^{(r\mu_i/d)-1}}{\prod_{j=1}^m m_j(\mu)! \prod_{i=1}^{l(\mu)} (\frac{r\mu_i}{d})!} \\ &= \pi_{\frac{r}{d}\mu} \prod_{i=1}^{l(\mu)} \mu_i^{(r\mu_i/d)-1}, \end{aligned} \tag{34}$$

and

$$\begin{aligned} G_{(\frac{d}{\mu_i})^{r\mu_i/d}}\big(t^{\mu_i}\big) &= \big(tA_{(r\mu_i/d)-1}(t)[d/\mu_i]_t^{r\mu_i/d}\big)_{d/\mu_i}\Big|_{t=t^{\mu_i}} \\ &= \big(t^{\mu_i} A_{(r\mu_i/d)-1}\big(t^{\mu_i}\big)[d/\mu_i]_{t^{\mu_i}}^{r\mu_i/d}\big)_{d,\mu_i}. \end{aligned} \tag{35}$$

We now have

$$\begin{aligned} \sum_{j=0}^{n-1} \chi_{d^k}^{r^m,j} t^j &= \sum_{\mu \in \mathrm{Par}(m;d,r)} \pi_{\frac{r}{d}\mu} \prod_{i=1}^{l(\mu)} \mu_i^{\frac{r}{d}\mu_i - 1} \big(t^{\mu_i} A_{\frac{r}{d}\mu_i - 1}\big(t^{\mu_i}\big)[d/\mu_i]_{t^{\mu_i}}^{\frac{r}{d}\mu_i}\big)_{d,\mu_i} \\ &= \sum_{\mu \in \mathrm{Par}(m;d,r)} \pi_{\frac{r}{d}\mu} \prod_{i=1}^{l(\mu)} \big(tA_{\frac{r}{d}\mu_i - 1}(t)[d]_t^{\frac{r}{d}\mu_i}\big)_{d,\mu_i}, \end{aligned}$$

the first equality being obtained by substituting (34) and (35) into (33), and the second following from Lemma 5.1. □

We use Theorem 5.2 to handle general λ.

Theorem 5.3. *Say $\lambda \in \mathrm{Par}(n)$ and $n = kd$.*

1. *If there is some $r \in [n]$ such that d does not divide $rm_r(\lambda)$ then $\chi_{d^k}^{\lambda,j} = 0$ for all $0 \leqslant j \leqslant n-1$.*

2. *If d divides $rm_r(\lambda)$ for all $r \in [n]$ then*

$$\sum_{j=0}^{n-1} \chi_{d^k}^{\lambda,j} t^j = \binom{\frac{n}{d}}{\frac{1m_1(\lambda)}{d}, \frac{2m_2(\lambda)}{d}, \ldots, \frac{nm_n(\lambda)}{d}} \prod_{r=1}^{n} \sum_{j=0}^{rm_r(\lambda)-1} \chi_{d^{rm_r(\lambda)/d}}^{r^{m_r(\lambda)},j} t^j.$$

Proof. It follows directly from (13) that

$$\sum_{j=0}^{n-1} Q_{\lambda,j} t^j = \prod_{r=1}^{n} \sum_{j=0}^{rm_r(\lambda)-1} Q_{r^{m_r(\lambda)},j} t^j. \tag{36}$$

Expressing both sides of (36) in terms of the power sum basis, we get

$$\sum_{\mu \in \mathrm{Par}(n)} z_\mu^{-1} \sum_{j=0}^{n-1} \chi_\mu^{\lambda,j} t^j \mathrm{p}_\mu = \prod_{r=1}^{n} \sum_{\nu \in \mathrm{Par}(rm_r(\lambda))} z_\nu^{-1} \sum_{j=0}^{rm_r(\lambda)-1} \chi_\nu^{r^{m_r(\lambda)},j} t^j \mathrm{p}_\nu. \tag{37}$$

Equating coefficients of p_{d^k} in (37) we see that if d does not divide every $rm_r(\lambda)$ then $\chi_{d^k}^{\lambda,j} = 0$ for all j, while if d divides every $rm_r(\lambda)$ then

$$\begin{aligned}
\sum_{j=0}^{n-1} \chi_{d^k}^{\lambda,j} t^j &= z_{d^k} \prod_{r=1}^{n} z_{d^{rm_r(\lambda)/d}}^{-1} \sum_{j=0}^{rm_r(\lambda)-1} \chi_{d^{rm_r(\lambda)/d}}^{r^{m_r(\lambda)},j} t^j \\
&= \frac{d^{n/d}(n/d)!}{\prod_{r=1}^{n} d^{rm_r(\lambda)/d}(rm_r(\lambda)/d)!} \prod_{r=1}^{n} \sum_{j=0}^{rm_r(\lambda)-1} \chi_{d^{rm_r(\lambda)/d}}^{r^{m_r(\lambda)},j} t^j \\
&= \binom{\frac{n}{d}}{\frac{1m_1(\lambda)}{d}, \frac{2m_2(\lambda)}{d}, \ldots, \frac{nm_n(\lambda)}{d}} \prod_{r=1}^{n} \sum_{j=0}^{rm_r(\lambda)-1} \chi_{d^{rm_r(\lambda)/d}}^{r^{m_r(\lambda)},j} t^j. \qquad \square
\end{aligned}$$

5.2. *The permutation character $\theta^{\lambda,j}$ of G_n*

Note that, upon considering cycle notation for elements of S_n, it is straightforward to show that if $\sigma \in S_{\lambda,j}$ then $\gamma_n^{-1}\sigma\gamma_n \in S_{\lambda,j}$. Thus the claim in Theorem 1.2 that G_n acts on $S_{\lambda,j}$ is correct. Let $\theta^{\lambda,j}$ denote the permutation character of the action of G_n on $S_{\lambda,j}$. Hence, $\theta^{\lambda,j}(\tau)$ is the number of elements of $S_{\lambda,j}$ centralized by $\tau \in G_n$. For $\nu \in \mathrm{Par}(n)$, let $\theta_\nu^{\lambda,j} = \theta^{\lambda,j}(\tau)$, where τ is any permutation of cycle type ν. Since all $\tau \in G_n$ have cycle type of the form d^k, where $dk = n$, we need only concern ourselves with $\nu = d^k$.

With Theorems 5.2 and 5.3 in hand, we now produce matching results for the permutation characters $\theta^{\lambda,j}$ of G_n. Again we begin with the case where $\lambda = r^m$ for some divisor r of n. Before doing so, we derive, in the form most useful for our arguments, some known facts about centralizers in S_n of elements of G_n, along with straightforward consequences of these facts.

Fix positive integers n, k, d with $n = kd$. Set $\tau = \gamma_n^{-k} \in G_n$. Note that $C_{S_n}(\tau) = C_{S_n}(\gamma_n^k)$. For $i \in [n]$, we have

$$i\tau = \begin{cases} i-k, & i > k, \\ i-k+n, & i \leqslant k. \end{cases}$$

Now τ has cycle type d^k, and we can write τ as the product of k d-cycles, $\tau = \tau_1 \dots \tau_k$, where τ_i has support

$$X_i := \{ j \in [n]:\ j \equiv i \bmod k \}.$$

It follows that if $\sigma \in C_{S_n}(\tau)$ then for each $i \in [k]$ there is some $j \in [k]$ such that $X_i\sigma = X_j$. Thus we have an action of $C_{S_n}(\tau)$ on $\{X_1, \dots, X_k\}$, which gives rise to a homomorphism

$$\Phi : C_{S_n}(\tau) \to S_k.$$

Given $\rho \in S_k$, define $\hat{\rho} \in S_n$ to be the element that, for $r \in [k]$ and $q \in \{0, \dots, d-1\}$, maps $r + qk$ to $r\rho + qk$. It is straightforward to check that $\hat{\rho} \in C_{S_n}(\tau)$ and $\Phi(\hat{\rho}) = \rho$. Moreover, if we set

$$R := \{\hat{\rho}:\ \rho \in S_k\},$$

then $R \leqslant C_{S_n}(\tau)$ and the restriction of Φ to R is an isomorphism. It follows that if we set $K = \text{kernel}(\Phi)$ then $C_{S_n}(\tau)$ is the semidirect product of K and R. Now

$$K = \{ \sigma \in C_{S_n}(\tau):\ X_i\sigma = X_i \text{ for all } i \in [k] \} = \prod_{i=1}^{k} C_{S_{X_i}}(\tau_i).$$

Since every d-cycle in S_d generates its own centralizer in S_d, we have

$$K = \prod_{i=1}^{k} \langle \tau_i \rangle = \left\{ \prod_{i=1}^{k} \tau_i^{e_i}:\ e_1, \dots, e_k \in \{0, \dots, d-1\} \right\}.$$

Now, given $\rho \in S_k$ and $e_1, \dots, e_k \in \{0, \dots, d-1\}$, set

$$\sigma := \tau_1^{e_1} \dots \tau_k^{e_k} \hat{\rho} \in C_{S_n}(\tau).$$

For $r \in [k]$ and $q \in \{0, \dots, d-1\}$, we have (with σ acting on the right)

$$(r + qk)\sigma = \begin{cases} r\rho + (q - e_r)k, & q \geqslant e_r, \\ r\rho + (q - e_r)k + n, & q < e_r. \end{cases} \tag{38}$$

It follows that $r + qk \in \text{Exc}(\sigma)$ if and only if either $q < e_r$ or $e_r = 0$ and $r < r\rho$. We collect in the next lemma some useful consequences of what we have just seen.

Lemma 5.4. *Let $n = dk$ and let $\tau = \gamma_n^{-k}$. Let $\sigma \in C_{S_n}(\tau)$. Then there exist unique $\rho \in S_k$ and $e_1, \dots, e_k \in \{0, \dots, d-1\}$ such that*

$$\sigma = \tau_1^{e_1} \dots \tau_k^{e_k} \hat{\rho},$$

and if we define E_0 to be the number of $r \in [k]$ such that $e_r = 0$ and $r \in \text{Exc}(\rho)$, then

$$\text{exc}(\sigma) = dE_0 + \sum_{i=1}^{k} e_i. \tag{39}$$

Note that the unique $\rho \in S_k$ of Lemma 5.4 is equal to $\Phi(\sigma)$ defined above. For $\mu \in \mathrm{Par}(k)$ and any divisor r of n, set

$$\mathsf{C}_\mu := \{\sigma \in C_{S_n}(\tau)\colon\ \Phi(\sigma) \in S_\mu\}$$

and

$$\mathsf{C}_{\mu,r} := \mathsf{C}_\mu \cap S_{r^{n/r}},$$

so $\mathsf{C}_{\mu,r}$ consists of those $\sigma \in C_{S_n}(\tau)$ such that σ has cycle type $r^{n/r}$ and $\Phi(\sigma)$ has cycle type μ.

Lemma 5.5. *For any divisor r of n, we have*

$$\sum_{\sigma \in \mathsf{C}_{(k),r}} t^{\mathrm{exc}(\sigma)} = \begin{cases} (tA_{k-1}(t)[d]_t^k)_{d,\frac{n}{r}}, & \text{if } k|r, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We begin by showing that

$$\mathsf{C}_{(k)} = \biguplus_{k|r|n} \mathsf{C}_{(k),r}. \tag{40}$$

Certainly the union on the right side of (40) is contained in $\mathsf{C}_{(k)}$, so we prove that this union contains $\mathsf{C}_{(k)}$. Let $\sigma \in \mathsf{C}_{(k)}$. By Lemma 5.4, we have

$$\sigma = \tau_1^{e_1} \dots \tau_k^{e_k} \hat{\rho}$$

for unique $\rho \in S_{(k)}$ and $e_1, \dots, e_k \in \{0, \dots, d-1\}$. It follows from (38) that for each $j \in [n]$ we have

$$j\sigma^k \equiv j - k\sum_{i=1}^{k} e_i \bmod n. \tag{41}$$

Moreover, if $j\sigma^l \equiv j \bmod k$ then $k|l$. Hence each cycle length in the cycle decomposition of σ is a multiple of k.

We claim that all cycles in the cycle decomposition of σ have length sk, where s is the order of $k\sum_{i=1}^k e_i$ in $\mathbb{Z}_n$. Indeed, it follows from (41) that for all $j \in [n]$,

$$j\sigma^{sk} \equiv j - sk\sum_{i=1}^{k} e_i \equiv j \bmod n,$$

which implies that $j\sigma^{sk} = j$. Hence the order of σ in S_n divides sk. It follows that every cycle length in the cycle decomposition of σ divides sk. Now we need only show that sk divides the length of each cycle. Suppose α is a cycle of length r and j is an element in the support of α. We have $k|r$ since k divides the length of every cycle. Again using (41) we have,

$$j = j\sigma^r = j(\sigma^k)^{r/k} \equiv j - \frac{r}{k}k\sum_{i=1}^{k} e_i \bmod n,$$

which implies that $(r/k)k\sum_{i=1}^{k} e_i \equiv 0 \bmod n$. Thus s, the order of $k\sum_{i=1}^{k} e_i$, divides r/k, which implies that sk divides r. We have therefore shown that sk divides the length of every cycle, and since we have already shown that every cycle length divides sk, we conclude that all cycles in the cycle decomposition of σ have the same length sk, that is, $\sigma \in \mathrm{C}_{(k),r}$ for some r satisfying $k|r|n$, as claimed in (40).

We have also shown that $\mathrm{C}_{(k),r} = \emptyset$ if k does not divide r. Thus the claim of the lemma holds when k does not divide r.

Next we show that if $\sigma \in \mathrm{C}_{(k),r}$ then

$$\gcd\bigl(\operatorname{exc}(\sigma), d\bigr) = \frac{n}{r}. \tag{42}$$

As above, write $\sigma = \tau_1^{e_1} \ldots \tau_k^{e_k} \hat{\rho}$. Since $d|n$, it follows from (39) that

$$\gcd\bigl(\operatorname{exc}(\sigma), d\bigr) = \gcd\left(\sum_{i=1}^{k} e_i, d\right). \tag{43}$$

Since $k\sum_{i=1}^{k} e_i$ has order r/k in $(\mathbb{Z}_n, +)$, we have that

$$\frac{r}{k} = \frac{n}{\gcd(n, k\sum_{i=1}^{k} e_i)} = \frac{d}{\gcd(d, \sum_{i=1}^{k} e_i)},$$

the first equality following from simple facts about modular arithmetic and the second from the fact that $n = dk$. Now we have

$$\gcd\left(d, \sum_{i=1}^{k} e_i\right) = \frac{kd}{r}, \tag{44}$$

and combining (44) with (43) gives (42).

Combining (40) and (42), we get

$$\sum_{\sigma \in \mathrm{C}_{(k),r}} t^{\operatorname{exc}(\sigma)} = \left(\sum_{\sigma \in \mathrm{C}_{(k)}} t^{\operatorname{exc}(\sigma)}\right)_{d, \frac{n}{r}} \tag{45}$$

for each divisor r of n.

For $\rho \in S_k$ and $i \in [k]$, set

$$f_{\rho,i}(t) := \begin{cases} t[d]_t & \text{if } i \in \operatorname{Exc}(\rho), \\ [d]_t & \text{otherwise.} \end{cases}$$

Then

$$\sum_{\sigma \in \Phi^{-1}(\rho)} t^{\operatorname{exc}(\sigma)} = \prod_{i=1}^{k} f_{\rho,i}(t) = t^{\operatorname{exc}(\rho)}[d]_t^k,$$

the first equality following from Lemma 5.4. It follows now from (25) that

$$\sum_{\sigma \in \mathsf{C}_{(k)}} t^{\operatorname{exc}(\sigma)} = tA_{k-1}(t)[d]_t^k, \tag{46}$$

and combining (45) and (46) yields the lemma. □

Lemma 5.6. *For divisors k, r of n and $\mu \in \operatorname{Par}(k)$, we have*

$$\sum_{\sigma \in \mathsf{C}_{\mu,r}} t^{\operatorname{exc}(\sigma)} = \pi_\mu \prod_{i=1}^{l(\mu)} \sum_{\sigma \in \mathsf{C}_{(\mu_i),r}} t^{\operatorname{exc}(\sigma)},$$

where π_μ is defined as in (30)*.*

Proof. Given $\sigma \in \mathsf{C}_{\mu,r}$, we write as usual $\sigma = \tau_1^{e_1} \dots \tau_k^{e_k} \hat{\rho}$. Now $\rho \in S_k$ has cycle type μ, so we can write $\rho = \rho_1 \dots \rho_{l(\mu)}$ as a product of disjoint cycles whose lengths form the partition μ. For $i \in [l(\mu)]$, let B_i be the support of ρ_i. We may assume that $|B_i| = \mu_i$ for all i. Set

$$\beta(\sigma) := \{B_1, \dots, B_{l(\mu)}\},$$

so $\beta(\sigma)$ is a partition of $[k]$. For $i \in [l(\mu)]$, set

$$\sigma_i := \left(\prod_{j \in B_i} \tau_j^{e_j}\right) \hat{\rho}_i \in S_n.$$

The supports of both $\hat{\rho}_i$ and $\prod_{j \in B_i} \tau_j^{e_j}$ are contained in

$$\hat{B}_i := \{j + qk\colon\ j \in B_i,\ 0 \leqslant q \leqslant d-1\}.$$

It follows that $\hat{\rho}_i$ and $\Pi_{j \in B_h} \tau_j^{e_j}$ commute for all $i \neq h$, so

$$\sigma = \prod_{i=1}^{l(\mu)} \sigma_i.$$

Moreover,

$$\operatorname{exc}(\sigma) = \sum_{i=1}^{l(\mu)} \operatorname{exc}(\sigma_i). \tag{47}$$

For $i \in [l(\mu)]$, define f_i to be the unique order preserving bijection from $\hat{B}_i$ to $[d\mu_i]$, and set

$$\overline{\sigma}_i := f_i^{-1} \sigma_i f_i.$$

Then, for each $i \in [l(\mu)]$, we have $\overline{\sigma}_i \in \mathsf{C}_{(\mu_i),r}$ and

$$\operatorname{exc}(\overline{\sigma}_i) = \operatorname{exc}(\sigma_i). \tag{48}$$

Let Π_μ be the set of partitions of $[k]$ that have $m_j(\mu)$ blocks of size j for each j. For each partition $X \in \Pi_\mu$, set

$$\mathrm{C}_X := \{\sigma \in \mathrm{C}_{\mu,r}\colon\ \beta(\sigma) = X\}.$$

The map from C_X to $\prod_{i=1}^{l(\mu)} \mathrm{C}_{(\mu_i),r}$ sending σ to $(\overline{\sigma}_1, \dots, \overline{\sigma}_{l(\mu)})$ is a bijection. Given (47) and (48), we see that

$$\sum_{\sigma \in \mathrm{C}_{\mu,r}} t^{\mathrm{exc}(\sigma)} = \sum_{X \in \Pi_\mu} \sum_{\sigma \in \mathrm{C}_X} t^{\mathrm{exc}(\sigma)} = |\Pi_\mu| \prod_{i=1}^{l(\mu)} \sum_{\rho \in \mathrm{C}_{(\mu_i),r}} t^{\mathrm{exc}(\rho)}.$$

It is straightforward to see that $|\Pi_\mu| = \pi_\mu$, so the lemma follows. □

Theorem 5.7. *Let $n, r, m, d, k \in \mathbb{P}$ with $n = rm = dk$. As in Theorem 5.2, let*

$$\mathrm{Par}(m; d, r) = \{\mu = (\mu_1, \dots, \mu_{l(\mu)}) \in \mathrm{Par}(m)\colon\ \mu_i | d | r\mu_i \text{ for all } i \in [l(\mu)]\}.$$

Then

$$\sum_{j=0}^{n-1} \theta_{d^k}^{r^m, j} t^j = \sum_{\mu \in \mathrm{Par}(m; d, r)} \pi_{\frac{r}{d}\mu} \prod_{i=1}^{l(\mu)} \left(t A_{\frac{r}{d}\mu_i - 1}(t)[d]_t^{\frac{r}{d}\mu_i}\right)_{d, \mu_i}. \tag{49}$$

Proof. We have

$$\begin{aligned}
\sum_{j=0}^{n-1} \theta_{d^k}^{r^m, j} t^j &= \sum_{\sigma \in C_{S_n}(\gamma_n^k) \cap S_{r^m}} t^{\mathrm{exc}(\sigma)} \\
&= \sum_{\mu \in \mathrm{Par}(k)} \sum_{\sigma \in \mathrm{C}_{\mu,r}} t^{\mathrm{exc}(\sigma)} \\
&= \sum_{\mu \in \mathrm{Par}(k)} \pi_\mu \prod_{i=1}^{l(\mu)} \sum_{\sigma \in \mathrm{C}_{(\mu_i),r}} t^{\mathrm{exc}(\sigma)} \\
&= \sum_{\mu \in \mathrm{Par}(k; r, d)} \pi_\mu \prod_{i=1}^{l(\mu)} \left(t A_{\mu_i - 1}(t)[d]_t^{\mu_i}\right)_{d, d\mu_i/r}.
\end{aligned}$$

Indeed, the first two equalities follow immediately from the definitions of $\theta_{d^k}^{r^m, j}$ and $\mathrm{C}_{\mu,r}$, respectively, while the third follows from Lemma 5.6 and the fourth from Lemma 5.5.

Now for $\mu \in \mathrm{Par}(k; r, d)$, set $\nu := \nu(\mu) := \frac{d}{r}\mu$, so $\mu = \frac{r}{d}\nu$. Now $\frac{d}{r}k = m$ and, since $\mu_i | r | d\mu_i$, we have $\nu_i | d | r\nu_i$ for all i. Thus $\nu \in \mathrm{Par}(m; d, r)$. From this we see that the map $\mu \mapsto \nu(\mu)$ is a bijection from $\mathrm{Par}(k; r, d)$ to $\mathrm{Par}(m; d, r)$. Thus we have

$$\sum_{j=0}^{n-1}\theta_{d^k}^{r^m,j}t^j=\sum_{\nu\in\mathrm{Par}(m;d,r)}\pi_{\frac{r}{d}\nu}\prod_{i=1}^{l(\nu)}\bigl(tA_{\frac{r}{d}\nu_i-1}(t)[d]_t^{\frac{r}{d}\nu_i}\bigr)_{d,\nu_i}$$

as claimed. □

Theorem 5.8. *Say $\lambda\in\mathrm{Par}(n)$ and $n=kd$.*

(1) *If there is some $r\in[n]$ such that d does not divide $rm_r(\lambda)$ then $\theta_{d^k}^{\lambda,j}=0$ for all $0\leqslant j\leqslant n-1$.*
(2) *If d divides $rm_r(\lambda)$ for all $r\in[n]$ then*

$$\sum_{j=0}^{n-1}\theta_{d^k}^{\lambda,j}t^j=\binom{\frac{n}{d}}{\frac{1m_1(\lambda)}{d},\frac{2m_2(\lambda)}{d},\ldots,\frac{nm_n(\lambda)}{d}}\prod_{r=1}^{n}\sum_{j=0}^{rm_r(\lambda)-1}\theta_{d^{rm_r(\lambda)/d}}^{r^{m_r(\lambda)},j}t^j.$$

Proof. Let $\sigma\in S_\lambda$, so σ can be written as a product of disjoint cycles in which there appear exactly $m_r(\lambda)$ r-cycles for each $r\in[n]$. For each such r, let σ_r be the product of all these $m_r(\lambda)$ r-cycles, and let B_r be the support of σ_r. If $\sigma\in C_{S_n}(\gamma_n^k)$ then γ_n^k commutes with each σ_r. It follows that γ_n^k stabilizes each B_r setwise. Therefore, for each $r\in[n]$, there is some $Y_r\subseteq[k]$ such that

$$B_r=\biguplus_{i\in Y_r}X_i,$$

where $X_i=\{h\in[n]:\ h\equiv i \bmod k\}$. Since $|X_i|=d$ for all i, it follows that $rm_r(\lambda)=|B_r|=d|Y_r|$, so (1) holds.

For each $r\in[n]$, let $\beta_r\in S_{B_r}$ act as γ_n^k does on B_r. We have $\gamma_n^k=\prod_{r=1}^n\beta_r$, and β_r commutes with σ_r for all r. For each $r\in[n]$, let f_r be the unique order preserving bijection from B_r to $[rm_r(\lambda)]$. Direct calculation shows that for each r, we have

$$f_r^{-1}\beta_rf_r=\gamma_{rm_r(\lambda)}^{rm_r(\lambda)/d}. \tag{50}$$

Also,

$$\mathrm{exc}(\sigma)=\sum_{r=1}^n\mathrm{exc}(\sigma_r)=\sum_{r=1}^n\mathrm{exc}\bigl(f_r^{-1}\sigma_rf_r\bigr). \tag{51}$$

On the other hand suppose we are given an ordered n-tuple $(Y_1,\ldots,Y_n)$ of subsets of $[k]$ such that

(a) $|Y_r|=rm_r(\lambda)/d$ for each $r\in[n]$, and
(b) $[k]=\biguplus_{r=1}^nY_r$,

and we set $B_r=\biguplus_{i\in Y_r}X_i$ for each r. Then each B_r is γ_n^k-invariant, and if we set β_r equal to the restriction of γ_n^k to B_r, we can obtain $\sigma\in C_{S_n}(\gamma_n^k)\cap S_\lambda$ by choosing, for each r, any $\sigma_r\in S_{B_r}$ of type $r^{m_r(\lambda)}$ commuting with β_r and setting $\sigma=\prod_{r=1}^n\sigma_r$. The number of n-tuples satisfying (a) and (b) is

$$\binom{\frac{n}{d}}{\frac{1m_1(\lambda)}{d},\frac{2m_2(\lambda)}{d},\ldots,\frac{nm_n(\lambda)}{d}},$$

and the theorem now follows from (50) and (51). □

Comparing Theorem 5.7 with Theorem 5.2 and then comparing Theorem 5.8 with Theorem 5.3, we obtain

$$\chi_{d^k}^{Q_{\lambda,j}} = \theta_{d^k}^{\lambda,j}$$

for all $\lambda \in \mathrm{Par}(n)$, $j \in \{0, 1, \ldots, n-1\}$, and $d, k \in \mathbb{P}$ such that $dk = n$. Theorem 1.2 now follows from Proposition 3.2.

6. Some additional results

As mentioned in the introduction, Theorem 1.2 is a refinement of (5). In this section we show that the less refined result can also be obtained as a consequence of [16, Corollary 4.3], which states that

$$A_n^{\mathrm{maj},\mathrm{exc},\mathrm{fix}}(q,t,s) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{\substack{k_0 \geqslant 0 \\ k_1,\ldots,k_m \geqslant 2 \\ \sum k_i = n}} \begin{bmatrix} n \\ k_0, \ldots, k_m \end{bmatrix}_q s^{k_0} \prod_{i=1}^{m} tq[k_i - 1]_{tq}. \tag{52}$$

Although the alternative proof does not directly involve the Eulerian quasisymmetric functions, the proof of (52) given in [16] does. Hence the Eulerian quasisymmetric functions play an indirect role. In this section we also prove the identities (7) and (8) mentioned in the introduction and as a consequence of (8) obtain another cyclic sieving result.

Theorem 6.1. *Let $dk = n$. Then the following expressions are all equal.*

(i) $A_n^{\mathrm{maj},\mathrm{exc},\mathrm{fix}}(\omega_d, t\omega_d^{-1}, s)$,
(ii) $\sum_{\sigma \in C_{S_n}(\gamma_n^k)} t^{\mathrm{exc}(\sigma)} s^{\mathrm{fix}(\sigma)}$,
(iii) $A_k^{\mathrm{exc},\mathrm{fix}}(t, \frac{s^d + t[d-1]_t}{[d]_t})[d]_t^k$.

Proof. Let us prove first that (ii)=(iii). For $\rho \in S_k$ and $i \in [k]$ set

$$f_{\rho,i}(t,s) := \begin{cases} t[d]_t & \text{if } i \in \mathrm{Exc}(\rho), \\ s^d + t[d-1]_t & \text{if } i \in \mathrm{Fix}(\rho), \\ [d]_t & \text{otherwise.} \end{cases}$$

It follows from Lemma 5.4 that

$$\sum_{\sigma \in \Phi^{-1}(\rho)} t^{\mathrm{exc}(\sigma)} s^{\mathrm{fix}(\sigma)} = \prod_{i=1}^{k} f_{\rho,i}(t,s)$$
$$= t^{\mathrm{exc}(\rho)} [d]_t^{k - \mathrm{fix}(\rho)} \big(s^d + t[d-1]_t\big)^{\mathrm{fix}(\rho)}.$$

By summing over all $\rho \in S_k$, we obtain the equality of the expressions in (ii) and (iii).

Now we prove that (i)=(iii). By setting $q=1$ in (1) we obtain

$$\begin{aligned}1+\sum_{k\geqslant 1}A_k^{\mathrm{exc,fix}}\left(t,\frac{s^d+t[d-1]_t}{[d]_t}\right)[d]_t^k\frac{z^k}{k!}&=\frac{(1-t)e^{(s^d+t[d-1]_t)z}}{e^{t[d]_tz}-te^{[d]_tz}}\\&=\frac{(1-t)e^{s^dz}}{e^{(t[d]_t-t[d-1]_t)z}-te^{([d]_t-t[d-1]_t)z}}\\&=\frac{(1-t)e^{s^dz}}{e^{t^dz}-te^{z}}.\end{aligned}\tag{53}$$

It follows from (52) and Proposition 2.2 that

$$A_{dk}^{\mathrm{maj,exc,fix}}\left(\omega_d,t\omega_d^{-1},s\right)=\sum_{m\geqslant 0}\sum_{\substack{l_0\geqslant 0\\ l_1,\dots,l_m\geqslant 1\\ \sum l_i=k}}\binom{k}{l_0,\dots,l_m}s^{dl_0}\prod_{i=1}^m t[dl_i-1]_t.$$

Hence, by straightforward manipulation of formal power series we have,

$$\begin{aligned}1+\sum_{k\geqslant 1}A_{dk}^{\mathrm{maj,exc,fix}}\left(\omega_d,t\omega_d^{-1},s\right)\frac{z^k}{k!}&=\sum_{k\geqslant 0}\sum_{m\geqslant 0}\sum_{\substack{l_0\geqslant 0\\ l_1,\dots,l_m\geqslant 1\\ \sum l_i=k}}\binom{k}{l_0,\dots,l_m}s^{dl_0}\prod_{i=1}^m t[dl_i-1]_t\frac{z^k}{k!}\\&=e^{s^dz}\sum_{m,k\geqslant 0}\sum_{\substack{\sum l_i=k\\ l_1,\dots,l_m\geqslant 1}}\binom{k}{l_1,\dots,l_m}\prod_{i=1}^m t[dl_i-1]_t\frac{z^k}{k!}\\&=e^{s^dz}\sum_{m\geqslant 0}\sum_{l_1,\dots,l_m\geqslant 1}\prod_{i=1}^m t[dl_i-1]_t\frac{z^{l_i}}{l_i!}\\&=e^{s^dz}\sum_{m\geqslant 0}\left(\sum_{l\geqslant 1}t[dl-1]_t\frac{z^l}{l!}\right)^m.\end{aligned}$$

Further manipulation yields,

$$\begin{aligned}\sum_{m\geqslant 0}\left(\sum_{l\geqslant 1}t[dl-1]_t\frac{z^l}{l!}\right)^m&=\frac{1}{1-(\sum_{l\geqslant 1}t[dl-1]_t\frac{z^l}{l!})}\\&=\frac{1-t}{1-t+\sum_{l\geqslant 1}t(t^{dl-1}-1)\frac{z^l}{l!}}\\&=\frac{1-t}{1-t+e^{t^dz}-1-t(e^z-1)}\\&=\frac{1-t}{e^{t^dz}-te^z}.\end{aligned}$$

Hence

$$1+\sum_{k\geqslant 1} A_{dk}^{\mathrm{maj},\mathrm{exc},\mathrm{fix}}\left(\omega_d, t\omega_d^{-1}, s\right)\frac{z^k}{k!}=\frac{(1-t)e^{s^d z}}{e^{t^d z}-te^z}.$$

The result now follows from (53). □

Corollary 6.2. *Let $dk=n$. Then*

$$A_n^{\mathrm{maj},\mathrm{exc}}\left(\omega_d, t\omega_d^{-1}\right)=A_k(t)[d]_t^k.$$

Similar results hold for cycle-type q-Eulerian polynomials.

Theorem 6.3. *Let $dk=n$. Then*

$$A_{(n)}^{\mathrm{maj},\mathrm{exc}}\left(\omega_d, t\omega_d^{-1}\right)=\left(tA_{k-1}(t)\,[d]_t^k\right)_d \tag{54}$$

and

$$A_{(n+1)}^{\mathrm{maj},\mathrm{exc}}\left(\omega_d, t\omega_d^{-1}\right)=tA_k(t)\,[d]_t^k. \tag{55}$$

Proof. To prove (54), we apply the first equation of Proposition 3.2 which tells us that for all j, the coefficient of t^j in $A_{(n)}^{\mathrm{maj},\mathrm{exc}}(\omega_d, t\omega_d^{-1})$ is equal to $\chi_{d^k}^{Q_{(n),j}}$. By Theorem 4.1, $\chi_{d^k}^{Q_{(n),j}}$ equals the coefficient of t^j in $(tA_{k-1}(t)[d]_t^k)_d$, as $G_{d^k}(t)=(tA_{k-1}(t)[d]_t^k)_d$.

To prove (55), we apply the second equation of Proposition 3.2 which tells us that for all j, the coefficient of t^j in $A_{(n+1)}^{\mathrm{maj},\mathrm{exc}}(\omega_d, t\omega_d^{-1})$ is equal to $\chi_{1d^k}^{Q_{(n),j}}$. By Theorem 4.1, $\chi_{1d^k}^{Q_{(n),j}}$ equals the coefficient of t^j in $tA_k(t)[d]_t^k$, as $G_{1d^k}(t)=tA_k(t)[d]_t^k$. □

Corollary 6.4. *Let $S_{n,j}$ be the set of permutations in S_n with j excedances. Then the triple $(G_n, S_{n,j}, a_{(n+1),j+1}(q))$ exhibits the cyclic sieving phenomenon for all $j\in\{0,1,\dots,n-1\}$.*

Proof. That the triple exhibits the cyclic sieving phenomenon is equivalent to the equation

$$t\sum_{\sigma\in C_{S_n}(g)} t^{\mathrm{exc}(\sigma)}=A_{(n+1)}^{\mathrm{maj},\mathrm{exc}}\left(\omega_d, t\omega_d^{-1}\right),$$

for all divisors d of n and $g\in G_n$ of order d. This equation is a consequence of Theorems 6.1 and 6.3, which respectively say that the left side and the right side of the equation both equal $tA_k(t)[d]_t^{\frac{n}{d}}$. □

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