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Bounding quantities related to the packing density of $1(\ell+1)\ell\cdots 2$

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Abstract

We bound several quantities related to the packing density of the patterns $1(\ell+1)\ell\cdots 2$. These bounds sharpen results of Bóna, Sagan, and Vatter and give a new proof of the packing density of these patterns, originally computed by Stromquist in the case $\ell = 2$ and by Price for larger ℓ. We end with comments and conjectures.

1. Introduction

We say two sequences p, q of length n are of the same *type* if $p(i) < p(j)$ if and only if $q(i) < q(j)$ for all $i, j \in [n]$, that is, if p and q have the same pairwise comparisons. For an n-permutation p and an ℓ-permutation q we let $c_q(p)$ denote the number of l-subsequences of type q in p, and we say that p contains $c_q(p)$ copies of the *pattern* q. For example, 41523 contains exactly two 132-patterns, namely 152 and 153, so $c_{132}(41523) = 2$.

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We say that an n-permutation p is *q-optimal* if there is no n-permutation with more copies of q than p, and let

$$M_{n,q} = c_q(p) \quad \text{for a } q\text{-optimal } p.$$

Since there are a total of $\binom{n}{\ell}$ ℓ-subsequences in any n-permutation, we always have $0 \leqslant M_{n,q} \leqslant \binom{n}{|q|}$. The *packing density* of a permutation q is defined as

$$\delta(q) = \lim_{n\to\infty} \frac{M_{n,q}}{\binom{n}{|q|}}.$$

This limit exists because of the following theorem. An unpublished proof was given by Galvin and reproduced in Price's thesis. One can also find the demonstration in a paper of Albert, Atkinson, Handley, Holton, and Stromquist [1].

Theorem 1.1 [1,5]. *The ratio $M_{n,q}/\binom{n}{|q|}$ is weakly decreasing.*

Stromquist [6] computed the packing density of 132. Using similar techniques, Price computed the packing density of the patterns $q_\ell = 1(\ell+1)\ell\cdots 2$ for all $\ell \geqslant 2$. Information about the packing densities of other patterns can be found in Burstein et al. [3] and Hästö [4].

Theorem 1.2 [5]. *The packing density of q_ℓ is*

$$\beta = \ell\alpha(1-\alpha)^{\ell-1} \tag{1}$$

where α is the unique root of

$$f_\ell(x) = \ell x^{\ell+1} - (\ell+1)x + 1 \tag{2}$$

in the interval $(0,1)$.

Since $f_\ell(1/(\ell+1)) > 0$ and $f_\ell(1/\ell) < 0$, we have

$$\frac{1}{\ell+1} < \alpha < \frac{1}{\ell}. \tag{3}$$

The chart below shows approximate values of α and β for small ℓ.

ℓ	α	β
2	0.366	0.464
3	0.253	0.424
4	0.200	0.410
5	0.167	0.402

For the rest of the paper, we abbreviate M_{n,q_ℓ} to M_n. Price proved Theorem 1.2 by showing that

$$\frac{M_n}{\binom{n}{\ell+1}} = \beta + O\left(\frac{\log n}{n}\right).$$

We will reprove Theorem 1.2 by giving precise bounds on M_n.

Theorem 1.3. *For all $n \geqslant \ell \geqslant 2$,*

$$\beta\frac{(n-\ell)^{\ell+1}}{(\ell+1)!} \leqslant M_n \leqslant \beta\frac{(n+\delta_{2,\ell})^{\ell+1}}{(\ell+1)!},$$

where $\delta_{2,\ell}$ is the Kronecker delta (and not to be confused with the packing density $\beta = \delta(q_\ell)$).

Note that Theorem 1.2 follows immediately from the theorem just stated by merely dividing all sides by $\binom{n}{\ell+1}$ and taking $n \to \infty$. Also note that the lower bound follows from Price's calculation of the packing density of q_ℓ and the fact that $M_n/\binom{n}{\ell+1}$ is decreasing, but we will provide another demonstration in order to give a new proof of Theorem 1.2. We will also have other uses for the intermediate results needed to prove both bounds.

The rest of this paper is structured as follows. In the next section we give some preliminary definitions and previous results which will be needed for our bounds. Section 3 is devoted to proofs of bounds involving M_n. In the section following that, we provide bounds for a related quantity. Often our upper bound proofs from these sections will not work when $\ell = 2$, so Section 5 is devoted to a discussion of that case. Finally, we end with a section of comments and conjectures.

2. Definitions and previous results

We say that a permutation is *layered* if it is the concatenation of subwords (the *layers*) where the entries decrease within each layer, and increase between the layers. For example, 321548769 is a layered permutation with layers 321, 54, 876, and 9. The only permutations for which the packing density has been computed are layered or equivalent to layered permutations under one of the routine symmetries. The following theorem of Stromquist is crucial for computing these densities. Its proof may also be found in Price's thesis [5], and a generalization is proved in [1]. Bóna, Sagan, and Vatter [2] proved a similar result for n-permutations with $M_n - 1$ copies of q_ℓ, for any $l \geqslant 2$.

Theorem 2.1 [6]. *For all layered permutations q and positive integers n, there is a layered q-optimal n-permutation.*

Layered q_ℓ-optimal permutations have the following easily established recursive structure.

Proposition 2.2 [2]. *Let p be a layered q_ℓ-optimal n-permutation whose last layer is of length m. Then the leftmost $k = n - m$ elements of p form a q_ℓ-optimal k-permutation.*

The previous proposition implies that

$$M_n = \max_{1 \leqslant k < n} \left(M_k + k \binom{m}{\ell} \right). \tag{4}$$

The value of k that maximizes the right-hand side of (4) will be very important throughout this paper, so we give it a notation as follows.

Definition 2.3. For any positive integer $n > \ell$, let k_n denote the positive integer for which $M_k + k\binom{m}{\ell}$ is maximal. If there are several integers with this property, let k_n be the largest among them.

Once we have found the packing density of q_ℓ (Theorem 1.2), it is not hard to find the asymptotic behavior of k_n.

Corollary 2.4 [5]. *The limit of k_n/n is α.*

We will sharpen this result considerably in Section 4. We will also need some information about β. First are a couple of extremal expressions for β.

Lemma 2.5. *The quantity β satisfies*

$$\beta = \max_{0 \leqslant \gamma \leqslant 1} \frac{(\ell+1)\gamma(1-\gamma)^{\ell-1}}{1 + \gamma + \gamma^2 + \cdots + \gamma^\ell} = \min_{0 \leqslant \gamma \leqslant \alpha} \beta\gamma^\ell + (1-\gamma)^\ell.$$

In fact $\beta = \beta\alpha^\ell + (1-\alpha)^\ell$.

Proof. The maximum expression for β was given by Price [5] in his proof of Theorem 1.2.

After rearranging terms and plugging in the definition of β, proving the last equation is equivalent to showing that

$$\ell\alpha(1-\alpha)^{\ell-1}\left(\alpha^\ell - 1\right) + (1-\alpha)^\ell = 0.$$

Cancelling out $(1-\alpha)^{\ell-1}$ leaves the defining equation for α and thus proves the result.

Now to obtain the minimum expression, it suffices to show that $\beta\gamma^\ell + (1-\gamma)^\ell$ is an decreasing function of γ on the interval $[0, \alpha]$. It is an easy exercise in calculus to show that, in fact, it is decreasing on $[0, 1/\ell]$. So by (3) we are done. $\square$

In addition, we will need some upper bounds for β.

Lemma 2.6. *For all $\ell \geqslant 2$ we have*

$$\beta \leqslant \left(1 - \frac{1}{\ell}\right)^{\ell-1} \leqslant \frac{1}{2}.$$

Proof. For the first inequality, consider the function

$$f(x) = \ell x(1-x)^{\ell-1}. \tag{5}$$

Clearly $f(\alpha) = \beta$. Furthermore, elementary calculus shows that $f(x)$ is an increasing function on the interval $[0, 1/\ell]$, which contains α by (3). So $f(\alpha) \leqslant f(1/\ell)$ and we are done with the first bound. For the second inequality we use the usual bounds for alternating series to give

$$\left(1 - \frac{1}{\ell}\right)^{\ell-1} \leqslant 1 - \frac{\ell-1}{\ell} + \frac{(\ell-1)(\ell-2)}{2\ell^2} = \frac{1}{2} - \frac{1}{2\ell} + \frac{1}{\ell^2} \leqslant \frac{1}{2}$$

when $\ell \geqslant 2$. □

Bóna et al. gave crude bounds on k_n.

Proposition 2.7 [2]. *For $n > \ell$ we have*

$$\frac{n-\ell}{\ell+1} \leqslant k_n < \frac{n}{l}.$$

They also found that the sequence k_n is "continuous" in the following sense.

Theorem 2.8 (Continuity Theorem [2]). *The sequence $(k_n)_{n>\ell}$ diverges to infinity and satisfies*

$$k_{n-1} \leqslant k_n \leqslant k_{n-1} + 1$$

for all $n > l + 1$.

The Continuity Theorem will be very useful for us because it shows that there are only two possibilities for k_{n-1}: either k_n or $k_n - 1$.

Let $c_{n,i}$ denote the number of copies of q_ℓ in an n-permutation whose last layer is of length $n - i$ and whose leftmost i elements form a q_ℓ-optimal i-permutation. So for $1 \leqslant i < n$,

$$c_{n,i} = M_i + i\binom{n-i}{\ell}. \tag{6}$$

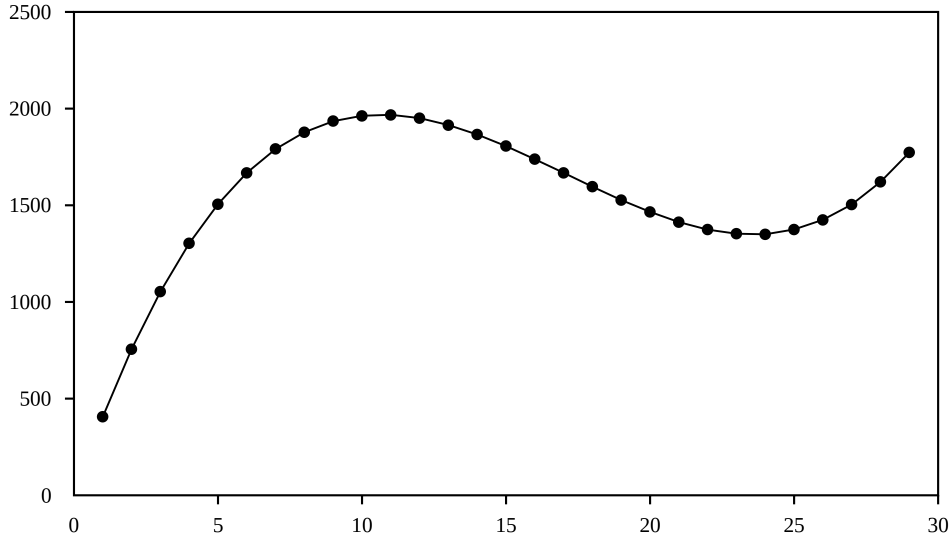

Fig. 1. A plot of the sequence $(c_{30,i})_{i=1}^{29}$ when $\ell = 2$. The sequence begins at $c_{30,1} = 406$ and rises to $c_{30,11} = M_{30} = 1968$, so $k_{30} = 11$. It then falls until $c_{30,24} = 1350$ before changing direction one last time.

As in [2], the sequences $(M_n)_{n\geqslant 1}$ and $(c_{n,i})_{i=1}^{n-1}$ will arise repeatedly, so we need to recall some results about them. We will frequently consider the difference $c_{n,i} - c_{n,i-1}$, so let us simplify it now

$$c_{n,i} - c_{n,i-1} = M_i - M_{i-1} + \frac{n-(\ell+1)i+1}{\ell}\binom{n-i}{\ell-1}. \tag{7}$$

We will also need the following result about differences of the M_n.

Lemma 2.9 [2]. *For all $n \geqslant 0$ we have*

$$0 \leqslant (M_{n+2} - M_{n+1}) - (M_{n+1} - M_n) \leqslant \binom{n}{\ell-1}.$$

To conclude our recap of results from [2], we state the Bimodal Theorem. It plays a crucial role in the arguments both in that paper and in this one.

Theorem 2.10 (Bimodal Theorem [2]). *For each positive integer $n > \ell$ there is some integer $j > n/\ell$ (depending, of course, on n) so that*:

(i) $c_{n,i-1} \leqslant c_{n,i}$ *if* $i \leqslant k_n$,
(ii) $c_{n,i-1} > c_{n,i}$ *if* $k_n < i \leqslant j$,
(iii) $c_{n,i-1} \leqslant c_{n,i}$ *if* $j < i < n$.

Figure 1 illustrates the phenomenon described by the Bimodal Theorem.

3. Bounds on M_n

For all $k \geqslant 1$, let n_k denote the least integer $n \geqslant \ell + 1$ such that $k_n = k$. As a trivial example, $n_1 = \ell + 1$. In general, we always have the following upper bound on n_k.

Proposition 3.1. *For all $k \geqslant 2$, we have*

$$n_k \leqslant (\ell + 1)k - 1.$$

Proof. Substituting $n = (\ell + 1)k - 1$ and $i = k$ reduces (7) to $c_{n,k} - c_{n,k-1} = M_k - M_{k-1} \geqslant 0$. By the Continuity Theorem, it suffices to show that $k_n \geqslant k$. Let j be as in the Bimodal Theorem. By that theorem we know that $\{c_{n,i}\}_{i=1}^{n-1}$ is bimodal with three sections $\{c_{n,i}\}_{i=1}^{k_n}$, $\{c_{n,i}\}_{i=k_n}^{j}$, and $\{c_{n,i}\}_{i=j}^{n-1}$, where the first and last sections are weakly increasing, while the second section is strictly decreasing. Therefore we must have either $k \leqslant k_n$, as desired, or $k > j$. However, $j > n/\ell = ((\ell + 1)k - 1)/\ell > k$ for $k \geqslant 2$, so the latter possibility cannot occur, finishing the proof. □

In the next lemma we compute n_k for all sufficiently small k.

Lemma 3.2. *For all $2 \leqslant k \leqslant \ell + 1$ we have*

$$n_k = (\ell + 1)k - 1.$$

Proof. Fix k between 2 and $\ell + 1$. Consider first the case when $2 \leqslant k \leqslant \ell$. Then $M_k = M_{k-1} = 0$ and, by the Continuity Theorem, we have $n_k - k > \ell - 1$ so $\binom{n_k - k}{\ell - 1} > 0$. We use (7) to get

$$0 \leqslant c_{n_k,k} - c_{n_k,k-1} = \frac{n_k - (\ell + 1)k + 1}{\ell} \binom{n_k - k}{\ell - 1},$$

which yields $n_k \geqslant (\ell + 1)k - 1$. The inequality in the other direction is given to us by Proposition 3.1, finishing this case.

Now consider $k = \ell + 1$. By Proposition 3.1 again, it suffices to show that $n_{\ell+1} \geqslant (\ell + 1)^2 - 1$. Using (7) again we get

$$c_{(\ell+1)^2-2,\ell+1} - c_{(\ell+1)^2-2,\ell} = 1 - \frac{1}{\ell} \binom{\ell(\ell + 1) - 2}{\ell - 1} < 0,$$

since

$$\frac{1}{\ell} \cdot \frac{\ell(\ell + 1) - 2}{\ell - 1} > 1$$

and the rest of the pairwise quotients in the binomial coefficient only make this term larger. Thus, by the Bimodal Theorem, the desired inequality for $n_{\ell+1}$ also holds. □

The next lemma will permit us to get preliminary bounds on k_n which will be needed to get the M_n bounds later in this section.

Lemma 3.3. *For each $k \geqslant 1$, the number of values of $n > \ell$ for which $k_n = k$ is at least ℓ and at most $\ell + 1$.*

Proof. We may assume $k \geqslant \ell + 1$ since smaller values have already been examined in the previous lemma.

We begin by showing that there are at least ℓ such values of n. Let $n = n_k$. So $c_{n-1,k} - c_{n-1,k-1} < 0$. Since $k_n \leqslant k_{n+1}$, we need only to establish that $c_{n+\ell-1,k+1} - c_{n+\ell-1,k} < 0$. Hence it suffices to show that

$$c_{n+\ell-1,k+1} - c_{n+\ell-1,k} \leqslant c_{n-1,k} - c_{n-1,k-1}.$$

Using (7) and Lemma 2.9 with $n = k - 1$, we see that the previous inequality will follow if we can show that

$$\begin{aligned}&\binom{k-1}{\ell-1} - \frac{1}{\ell}\binom{n+\ell-k-2}{\ell-1}\\&\quad\leqslant \frac{(\ell+1)k-1-(n-1)}{\ell}\left[\binom{n+\ell-k+2}{\ell-1} - \binom{n-k-1}{\ell-1}\right].\end{aligned}$$

The case $\ell = 2$ follows from straightforward computation, so we may assume $\ell \geqslant 3$ for the rest of this part of the proof. From Proposition 3.1,

$$\frac{(\ell+1)k-1-(n-1)}{\ell} \geqslant \frac{1}{\ell} > 0.$$

Because of this and the fact that $n+\ell-k+2 > n-k-1$, the last inequality in the previous paragraph will follow if we can show

$$\binom{k-1}{\ell-1} \leqslant \frac{1}{\ell}\binom{n+\ell-k-2}{\ell-1}.$$

This simplifies to

$$\ell(k-1)\cdots(k-\ell+1) \leqslant (n+\ell-k-2)\cdots(n-k).$$

For $\ell \geqslant 3$ there are sufficiently many factors on both sides of this inequality so that it will be proved if both

$$k-\ell+1 \leqslant n-k \tag{8}$$

and

$$\ell(k-1)(k-2) \leqslant (n+\ell-k-2)(n+\ell-k-3). \tag{9}$$

Both inequalities follow from the upper bound in Proposition 2.7 as follows. For (8) we have $2k-\ell+1<2k<\ell k<n$. For (9), note $\ell(k-1)(k-2)<n(k-2)$ while

$$\begin{aligned}(n+\ell-k-2)(n+\ell-k-3) &> (\ell k+\ell-k-2)(n+\ell-k-3)\\ &= \big((k+1)(\ell-1)-1\big)(n+\ell-k-3)\\ &\geqslant (2k-4)(n-k) > 2(k-2)\frac{n}{2},\end{aligned}$$

since $k<n/\ell<n/2$. This completes the proof that for all k there are at least ℓ values of n for which $k_n=k$.

We would now like to show that for all k there are at most $\ell+1$ values of n for which $k_n=k$. We do this by induction on k. Lemma 3.2 gives the result for $k\leqslant\ell$, so we may assume that $k>\ell$ and that the result is true for all values less than k. Let $n=n_k$, so $c_{n,k}-c_{n,k-1}\geqslant 0$. Then by the Continuity Theorem it suffices to show

$$c_{n+\ell+1,k+1}-c_{n+\ell+1,k}\geqslant 0$$

since that will imply that $k_{n+\ell+1}\geqslant k+1$. So it will be sufficient to show

$$c_{n+\ell+1,k+1}-c_{n+\ell+1,k}\geqslant c_{n,k}-c_{n,k-1}.$$

Using (7) and rearranging terms gives the equivalent inequality

$$(M_{k+1}-M_k)-(M_k-M_{k-1})\geqslant\frac{n-(\ell+1)k+1}{\ell}\left[\binom{n+\ell-k}{\ell-1}+\binom{n-k}{\ell-1}\right].$$

So by Lemma 2.9, it suffices to show

$$n-(\ell+1)k+1=n_k-(\ell+1)k+1\leqslant 0$$

and this is true by Proposition 3.1. □

Combining this result and Proposition 3.1 immediately gives an upper bound for n'_k which is defined as the *largest* value of n such that $k_n=k$. This will be important for our lower bound on k_n in the next section.

Corollary 3.4. *For all $k\geqslant 1$ we have*

$$n'_k\leqslant(\ell+1)k+\ell-1.$$

We will obtain better bounds on k_n in the next section by using our upcoming bounds on the difference M_n-M_{n-1}. But for the proof of the latter result we need a weaker upper bound which comes from Lemma 3.3.

Lemma 3.5. *For all $n \geqslant (\ell+1)\ell$ we have*

$$k_n \leqslant \frac{n-\ell}{\ell}.$$

Proof. Using Lemma 3.2, it is easy to see that this result holds for $(\ell+1)\ell \leqslant n \leqslant (\ell+1)^2 - 1$. To finish the demonstration, it suffices to prove the result for each n_k where $k > \ell + 1$. We may assume, by induction on k, that $k - 1 \leqslant (n_{k-1} - \ell)/\ell$. Now, by Lemma 3.3, $n_k \geqslant n_{k-1} + \ell$ which combines with the previous inequality to complete the proof. □

We will also need a technical corollary of the previous lemma.

Corollary 3.6. *For all $n \geqslant (\ell+1)\ell$ we have, with $k = k_n$,*

$$\beta\frac{(k-\ell+1)^\ell}{\ell!} + \binom{n-k-1}{\ell} \geqslant \beta\frac{k^\ell}{\ell!} + \frac{(n-k-\ell)^\ell}{\ell!}.$$

Proof. Rearranging terms and multiplying by $\ell!$, it suffices to show

$$(n-k-1)(n-k-2)\cdots(n-k-\ell) - (n-k-\ell)^\ell \geqslant \beta\big[k^\ell - (k-\ell+1)^\ell\big]. \quad (10)$$

Now using terminating approximations for positive and alternating series we have

$$(n-k-1)(n-k-2)\cdots(n-k-\ell) \geqslant (n-k-\ell)^\ell + \binom{\ell}{2}(n-k-\ell)^{\ell-1},$$

and

$$(k-\ell+1)^\ell \geqslant k^\ell - \ell(\ell-1)k^{\ell-1},$$

respectively. Comparing these with (10) reduces us to proving

$$(n-k-\ell)^{\ell-1}/2 \geqslant \beta k^{\ell-1}.$$

Lemma 2.6 gives us $\beta \leqslant 1/2$ so we will be done if $n - k - \ell \geqslant k$. But by the previous lemma, $k \leqslant (n-\ell)/\ell \leqslant (n-\ell)/2$ which is equivalent. □

We are now ready to prove one of our most useful results which gives bounds on the differences $M_n - M_{n-1}$. This will be used to get both our bounds on M_n in this section and our bounds on k_n in the next.

Theorem 3.7. *If $\ell \geqslant 3$ and $n \geqslant 1$, then we have*

$$M_n - M_{n-1} \leqslant \beta\frac{(n-1)^\ell}{\ell!}.$$

Furthermore, for all $\ell \geqslant 2$ and $n \geqslant \ell$,

$$M_n - M_{n-1} \geqslant \beta \frac{(n-\ell)^\ell}{\ell!}.$$

Proof. We begin by proving the upper bound by induction on n. For $n \leqslant \ell$ this bound is trivial, so we may assume that $n \geqslant \ell + 1$. So $k = k_n$ is well defined. Directly from the definitions

$$M_{n-1} \geqslant c_{n-1,k} = M_k + k\binom{n-k-1}{\ell}.$$

Combining this with (4) yields

$$M_n - M_{n-1} \leqslant k\binom{n-k-1}{\ell-1} \leqslant \frac{k(n-k-1)^{\ell-1}}{(\ell-1)!}. \tag{11}$$

Similarly,

$$M_{n-1} \geqslant c_{n-1,k-1} = M_{k-1} + (k-1)\binom{n-k}{\ell}.$$

Also $(n-k)(n-k-2) \leqslant (n-k-1)^2$, and because $\ell \geqslant 3$ there are enough factors in the binomial coefficient so that

$$M_n - M_{n-1} \leqslant M_k - M_{k-1} + \binom{n-k}{\ell} \leqslant M_k - M_{k-1} + \frac{(n-k-1)^\ell}{\ell!}. \tag{12}$$

Combining (11) and (12) we get

$$M_n - M_{n-1} \leqslant \gamma\left(M_k - M_{k-1} + \frac{(n-k-1)^\ell}{\ell!}\right) + (1-\gamma)\frac{k(n-k-1)^{\ell-1}}{(\ell-1)!} \tag{13}$$

for all $\gamma \in [0,1]$. By induction $M_k - M_{k-1} \leqslant \beta(k-1)^\ell/\ell! < \beta k^\ell/\ell!$. Making this substitution and setting $\gamma = k/(n-1)$ gives

$$\begin{aligned} M_n - M_{n-1} &\leqslant \gamma\left(\frac{\beta\gamma^\ell(n-1)^\ell}{\ell!} + \frac{(1-\gamma)^\ell(n-1)^\ell}{\ell!}\right) + (1-\gamma)\frac{\gamma(1-\gamma)^{\ell-1}(n-1)^\ell}{(\ell-1)!} \\ &= \left(\beta\gamma^{\ell+1} + (\ell+1)\gamma(1-\gamma)^\ell\right)\frac{(n-1)^\ell}{\ell!}. \end{aligned}$$

By Lemma 2.5, we know that for all $\gamma \in [0,1]$,

$$\frac{(\ell+1)\gamma(1-\gamma)^{\ell-1}}{1+\gamma+\gamma^2+\cdots+\gamma^\ell} \leqslant \beta,$$

so

$$(\ell+1)\gamma(1-\gamma)^{\ell} \leqslant \beta\bigl(1-\gamma^{\ell+1}\bigr).$$

It follows that

$$\beta\gamma^{\ell+1} + (\ell+1)\gamma(1-\gamma)^{\ell} \leqslant \beta,$$

completing the proof of the upper bound.

We will have to break the proof of the lower bound into two cases depending on the size of n.

First suppose that $n < (\ell+1)\ell$. By the Continuity Theorem we have two subcases depending upon whether $k_{n-1} = k-1$ or k. Suppose that the former is true so that we have, by Lemma 3.2, $k \leqslant \ell$. Then using $M_k = M_{k-1} = 0$ and (4) gives

$$M_n - M_{n-1} = k\binom{n-k}{\ell} - (k-1)\binom{n-k}{\ell} = \binom{n-k}{\ell}.$$

We would like to show that the right-hand side of this inequality is at least $\beta(n-\ell)^{\ell}/\ell!$. So by Lemma 2.6, it suffices to show that

$$\left(1-\frac{1}{\ell}\right)^{\ell-1}(n-\ell)^{\ell} \leqslant (n-k)(n-k-1)\cdots(n-k-l+1).$$

Note that since $k \leqslant \ell$ we have $n-\ell \leqslant n-k$ so we are reduced to proving

$$\left(1-\frac{1}{\ell}\right)^{\ell-1}(n-\ell)^{\ell-1} \leqslant (n-k-1)\cdots(n-k-l+1).$$

This last inequality will follow if we can show

$$\left(1-\frac{1}{\ell}\right)(n-\ell) \leqslant n-k-l+1. \tag{14}$$

But multiplying out the left-hand side and cancelling shows that this is true because of Proposition 2.7.

Now suppose that $k_{n-1} = k$. Then Lemma 3.2 implies that $k < \ell$ because of the bounds on n in this case. As before, we can compute

$$M_n - M_{n-1} = k\binom{n-k}{\ell} - k\binom{n-k-1}{\ell} = k\binom{n-k-1}{\ell-1}. \tag{15}$$

Using Lemma 2.6 again, we see that we need to prove

$$\left(1-\frac{1}{\ell}\right)^{\ell-1}(n-\ell)^{\ell} \leqslant \ell k(n-k-1)(n-k-2)\cdots(n-k-l+1).$$

But by Proposition 2.7 again

$$\ell^2 k > (\ell^2 - 1)k = (\ell - 1)(\ell + 1)k \geqslant (\ell - 1)(n - \ell) = \ell\left(1 - \frac{1}{\ell}\right)(n - \ell).$$

Furthermore, $k < \ell$ implies $n - k - 1 \geqslant n - \ell$ and (14) takes care of the remaining factors.

We may now assume that $n \geqslant (\ell + 1)\ell$. Again we have two subcases. If $k_{n-1} = k$, then $k \geqslant \ell$. Also (15) still holds and so

$$M_n - M_{n-1} \geqslant \frac{k(n - k - \ell)^{\ell-1}}{(\ell - 1)!} = \ell\gamma(1 - \gamma)^{\ell-1}\frac{(n - \ell)^\ell}{\ell!}, \tag{16}$$

where γ is defined by $k = \gamma(n - \ell)$. Note that $\gamma < 1/\ell$ by Lemma 3.5. Also, by our remarks about the function $f(x)$ of Eq. (5) in the proof of Lemma 2.6, we have the lower bound in the theorem as long as $\gamma \in [\alpha, 1/\ell)$.

To see what happens if $\gamma < \alpha$, we use the fact that $M_n \geqslant c_{n,k+1}$, Corollary 3.6, and induction to get

$$\begin{aligned} M_n - M_{n-1} &\geqslant M_{k+1} - M_k + \binom{n-k-1}{\ell} \geqslant \beta\frac{(k - \ell + 1)^\ell}{\ell!} + \binom{n-k-1}{\ell} \\ &\geqslant \beta\frac{k^\ell}{\ell!} + \frac{(n - k - \ell)^\ell}{\ell!} = \left(\beta\gamma^\ell + (1 - \gamma)^\ell\right)\frac{(n - \ell)^\ell}{\ell!}. \end{aligned}$$

But since $\gamma \in [0, \alpha)$, we can use Lemma 2.5 to conclude that our desired lower bound holds. So we are now done with the case where $k_{n-1} = k$.

Now assume that $k_{n-1} = k - 1$ so that $k > \ell$ because of the bound on n. Then from (4) we get that

$$M_n - M_{n-1} = M_k - M_{k-1} + \binom{n-k}{\ell}.$$

Thus the first bound in the previous string of inequalities holds with k replaced by $k - 1$. But $k - 1 \geqslant \ell$ so the same arguments used there apply to give the lower bound we seek. Similarly, since $M_n \geqslant c_{n,k-1}$ we can use (6) to get

$$M_n - M_{n-1} \geqslant (k - 1)\binom{n-k}{\ell-1},$$

which can be compared with (16) to complete the proof of this case and of the theorem itself. □

We are now in a position to take care of most of the cases in Theorem 1.3.

Theorem 3.8. *Suppose $n \geqslant \ell$. If $\ell \geqslant 3$, then we have*

$$M_n \leqslant \beta \frac{n^{\ell+1}}{(\ell+1)!}.$$

Furthermore, for all $\ell \geqslant 2$,

$$M_n \geqslant \beta \frac{(n-\ell)^{\ell+1}}{(\ell+1)!}.$$

Proof. Both bounds are trivial if $n = \ell$ since $M_\ell = 0$. So suppose $n > \ell$.

For the upper bound, we use the previous theorem and the standard way in which sums are used to bound integrals to get

$$M_n = \sum_{i=\ell+1}^{n} (M_i - M_{i-1}) \leqslant \frac{\beta}{\ell!} \sum_{i=\ell+1}^{n} (i-1)^\ell \leqslant \frac{\beta}{\ell!} \int_0^n x^\ell \,\mathrm{d}x = \frac{\beta}{(\ell+1)!} n^{\ell+1}.$$

There are two possible proofs of the lower bound at this point. Either one can mimic the demonstration of the upper bound or appeal to Theorems 1.1 and 1.2 to get

$$M_n \geqslant \beta \binom{n}{\ell+1} \geqslant \beta \frac{(n-\ell)^{\ell+1}}{(\ell+1)!}.$$

Using either technique, we are done. □

4. Bounds on k_n

We can now use the results of the previous section to supply bounds for k_n which will be a considerable improvement over those obtainable from Price's work. The best that can be gotten from Corollary 2.4 is $k_n = \alpha n + o(n)$. We will prove that in fact $k_n = \alpha n + O(1)$ with a constant inside the big oh that is less than 2.

Theorem 4.1. *For $\ell \geqslant 3$ and $n > \ell$ we have*

$$k_n \leqslant \alpha(n-\ell) + 1.$$

Proof. Let $k = k_n$. Note that it suffices to prove the bound when $n = n_k$.

Clearly the result is true for $k = 1$ and $n_1 = \ell + 1$. Next suppose that $2 \leqslant k \leqslant \ell + 1$. Then by Lemma 3.2, our desired inequality is equivalent to $k - 1 \leqslant (\ell+1)\alpha(k-1)$ which is true by (3).

For $k > \ell + 1$ we still have Proposition 3.1 which gives $n - (\ell+1)k + 1 \leqslant 0$. So by Theorem 3.7 and the fact that $\ell \geqslant 3$,

$$\begin{aligned}0 \leqslant c_{n,k} - c_{n,k-1} &= M_k - M_{k-1} + \frac{n-(\ell+1)k+1}{\ell}\binom{n-k}{\ell-1} \\ &\leqslant \beta\frac{(k-1)^\ell}{\ell!} + \frac{[n-\ell-(\ell+1)(k-1)][n-\ell-(k-1)]^{\ell-1}}{\ell!}.\end{aligned}$$

Define γ by $k-1=\gamma(n-\ell)$. So it suffices to show $\gamma \leqslant \alpha$. Clearly $\gamma \geqslant 0$ and since $k > \ell+1$ we can apply Lemma 3.5 to get $\gamma(n-\ell) < k \leqslant (n-\ell)/\ell$, so $\gamma < 1/\ell$. Rewriting the last expression in the previous paragraph in terms of γ and cancelling $\ell!$ gives

$$0 \leqslant \beta\gamma^\ell(n-\ell)^\ell + \bigl(1-(\ell+1)\gamma\bigr)(1-\gamma)^{\ell-1}(n-\ell)^\ell.$$

Since $n > \ell$ we have

$$0 \leqslant \beta\gamma^\ell + \bigl(1-(\ell+1)\gamma\bigr)(1-\gamma)^{\ell-1}.$$

Now define

$$g(x) = \beta x^\ell + \bigl(1-(\ell+1)x\bigr)(1-x)^{\ell-1},$$

so we have $g(\gamma) \geqslant 0$. Using the defining equations for α and β gives

$$g(\alpha) = \ell\alpha^{\ell+1}(1-\alpha)^{\ell-1} + \bigl(1-(\ell+1)\alpha\bigr)(1-\alpha)^{\ell-1} = 0.$$

This implies that

$$0 \leqslant g(\gamma) = g(\alpha) + \int_\alpha^\gamma g'(x)\,\mathrm{d}x = \int_\alpha^\gamma g'(x)\,\mathrm{d}x.$$

Since $0 \leqslant \alpha, \gamma \leqslant 1/\ell$, we can prove $\gamma \leqslant \alpha$ by showing that $g'(x) < 0$ on $[0, 1/\ell]$. Now

$$g'(x) = \beta\ell x^{\ell-1} - (\ell+1)(1-x)^{\ell-1} - \bigl(1-(\ell+1)x\bigr)(\ell-1)(1-x)^{\ell-2}.$$

So we want, after transposing terms,

$$\beta\ell x^{\ell-1} < (\ell+1)(1-x)^{\ell-1} + \bigl(1-(\ell+1)x\bigr)(\ell-1)(1-x)^{\ell-2}.$$

Taking the maximum of the left-hand side and the minimum of the right-hand side on the interval $[0, 1/\ell]$, it suffices to show that

$$\frac{\ell^2\alpha(1-\alpha)^{\ell-1}}{\ell^{\ell-1}} < \frac{(\ell+1)(\ell-1)^{\ell-1}}{\ell^{\ell-1}} - \frac{(\ell-1)^{\ell-1}}{\ell^{\ell-1}} = \frac{\ell(\ell-1)^{\ell-1}}{\ell^{\ell-1}}.$$

But $\ell^2\alpha(1-\alpha)^{\ell-1} < \ell(\ell-1)^{\ell-1}$ since by Eq. (3) we have both that $\ell^2\alpha < \ell$ and that $(1-\alpha)^{\ell-1} < (\ell-1)^{\ell-1}$. $\square$

We also have a lower bound with only a slightly larger constant.

Theorem 4.2. *For $\ell \geqslant 2$ and n sufficiently large,*

$$k_n \geqslant \alpha(n-\ell)-1.$$

Proof. Let $k=k_n$. Note that it suffices to prove the bound when $n=n'_k$. Note also that by Corollary 3.4 we have $n-(\ell+1)k-\ell<0$. Using this fact, the Bimodal Theorem, Eq. (7), and Theorem 3.7, we have

$$\begin{aligned} 0 > c_{n,k+1}-c_{n,k} &= M_{k+1}-M_k+\frac{n-(\ell+1)k-\ell}{\ell}\binom{n-k-1}{\ell-1} \\ &\geqslant \frac{\beta(k-\ell+1)^\ell+[n-(\ell+1)k-\ell](n-k-1)^{\ell-1}}{\ell!}. \end{aligned}$$

Now define γ by $k-\ell+1=\gamma(n-\ell)$. By taking n sufficiently large we can assume that $k+1\geqslant \ell$ and so $\gamma\geqslant 0$. Also, Theorem 4.1 implies that $\gamma(n-\ell)\leqslant\alpha(n-\ell)-\ell+2$ and so $\gamma\leqslant\alpha$. Substituting for γ to replace k in the last inequality of the previous paragraph we get, after multiplying by $\ell!/(n-\ell)^\ell$,

$$\beta\gamma^\ell+\left[1-\frac{\ell^2-1}{n-\ell}-(\ell+1)\gamma\right](1-\gamma)^{\ell-1}<0.$$

Let $\varepsilon=(\ell^2-1)/(n-\ell)$ and note that we can make ε as small a positive number as we wish by taking n large. Define a function

$$h(x)=\beta x^\ell+\bigl[1-\varepsilon-(\ell+1)x\bigr](1-x)^{\ell-1}$$

so that $h(\gamma)<0$. Using the defining equations for α and β, one can also compute that

$$h(\alpha)=-\varepsilon(1-\alpha)^{\ell-1}. \tag{17}$$

We want to mimic the integration trick used in the proof of the upper bound for k_n, so we need some information about $h'(x)$. First note that

$$\begin{aligned} h'(x) &= \ell\beta x^{\ell-1}-\bigl[(\ell+1)(1-x)+(\ell-1)(1-\varepsilon-(\ell+1)x)\bigr](1-x)^{\ell-2} \\ &= \ell\beta x^{\ell-1}-\bigl[(\ell+1)(1-\ell x)+(\ell-1)(1-\varepsilon)\bigr](1-x)^{\ell-2}. \end{aligned}$$

Taking one more derivative, one can see that $h''(x)\geqslant 0$ on $[0,1/\ell]$ as long as the factor in the final set of square brackets above is nonnegative. And this can be ensured by taking $1-\varepsilon>0$. So $h'(x)$ is increasing on this interval, and since $\gamma\leqslant\alpha<1/\ell$ we can write

$$0>h(\gamma)=h(\alpha)-\int_\gamma^\alpha h'(x)\,\mathrm{d}x\geqslant h(\alpha)-(\alpha-\gamma)h'(\alpha). \tag{18}$$

Next we claim that

$$h'(\alpha) \leqslant -(\ell-1)(1-\alpha)^{\ell-2}.$$

Using the second expression for $h'(x)$ and the definition of β, we see that it is sufficient to prove, after cancelling $(1-\alpha)^{\ell-2}$, that

$$\ell^2\alpha^\ell(1-\alpha) - (\ell+1)(1-\ell\alpha) - (\ell-1)(1-\varepsilon) \leqslant -(\ell-1).$$

Expanding the left-hand side and using $\ell\alpha^{\ell+1} = (\ell+1)\alpha - 1$ on the $-\ell^2\alpha^{\ell+1}$ term reduces this inequality, after massive cancellation, to

$$\ell^2\alpha^\ell + (\ell-1)\varepsilon \leqslant 1. \tag{19}$$

But this last equation is true for sufficiently small ε since, by (3),

$$\ell^2\alpha^\ell \leqslant \ell^2\alpha^2 < 1.$$

So we have proved the claim.

Now divide (18) by $h(\alpha)$ (which is negative by (17)) and use the claim as well as (3) again to get

$$0 < 1 - (\alpha-\gamma)\frac{h'(\alpha)}{h(\alpha)} \leqslant 1 - (\alpha-\gamma)\frac{n-\ell}{(\ell+1)(1-\alpha)} < 1 - (\alpha-\gamma)\frac{n-\ell}{\ell}.$$

Solving for γ in this last inequality and plugging into its defining equation gives

$$k = \gamma(n-\ell) + \ell - 1 \geqslant \left(\alpha - \frac{\ell}{n-\ell}\right)(n-\ell) + \ell - 1 = \alpha(n-\ell) - 1$$

as desired. $\square$

To give a feel for how good these bounds are, we prove the following corollary.

Corollary 4.3. *For $\ell \geqslant 3$ and $n > \ell$ we have*

$$k_n - \alpha n < 1/4.$$

For $\ell \geqslant 2$ and sufficiently large n we have

$$k_n - \alpha n > -2.$$

Proof. The lower bound follows immediately from the previous theorem and (3). For the upper bound, it is easy to show by taking second derivatives that $f_{\ell-1}(x) \geqslant f_\ell(x)$ on the interval $[0, 1/\ell]$. It follows that α is a decreasing function of ℓ. Furthermore, using (3) again shows that $|1 - \ell\alpha| < \alpha$. Combining these observations with Theorem 4.1 gives

$$k_n - \alpha n \leqslant 1 - \ell\alpha < \alpha.$$

But now we are done since $1 - \ell\alpha < 1/4$ when $\ell = 3$ and $\alpha < 1/4$ for $\ell \geqslant 4$. □

5. The upper bounds for $\ell = 2$

To complete the proof of Theorem 1.3 we must address the upper bound when $\ell = 2$. This result, as well as the upper bound on k_n in the previous section, depends on Theorem 3.7 where the restriction $\ell \geqslant 3$ first appeared. This is not an accident as that theorem is false for $\ell = 2$. For example, when $\ell = 2$ we have $M_{17} - M_{16} = 60$, but $\beta 16^2/2$ is approximately 59.405. Worse yet, our computer experiments have shown that this is not an isolated counterexample. However, a weaker upper bound is true.

Theorem 5.1. *For $\ell = 2$ and $n \geqslant \ell$ we have*

$$M_n - M_{n-1} \leqslant \beta \frac{n^2}{2}.$$

Proof. The proof is very similar to the demonstration of Theorem 3.7. There are only two changes. The first is that when bounding binomial coefficients one uses powers of $n - k$ rather than $n - k - 1$. Note that this removes the necessity to have $\ell \geqslant 3$. The other modification is that one substitutes $\gamma = k/n$. The rest of the proof proceeds as before. □

We can now obtain the $\ell = 2$ upper bound in Theorem 1.3. One uses the same proof as Theorem 3.8 but with the previous result taking the place of Theorem 3.7. Because of the similarity, we omit the details.

Theorem 5.2. *For $\ell = 2$ and $n \geqslant \ell$ we have*

$$M_n \leqslant \beta \frac{(n+1)^3}{3!}.$$

To obtain the bounds on k_n in this case, note that $f_\ell(x)$ always has $x = 1$ as a root. So dividing $f_2(x)$ by $x - 1$, we see that α must satisfy

$$\alpha^2 = \frac{-2\alpha + 1}{2}. \tag{20}$$

We can now plug this into the defining equation for β to get

$$\beta = 4\alpha - 1. \tag{21}$$

Theorem 5.3. *For $\ell = 2$ and all $n \geqslant 3$ we have*

$$k_n \leqslant \alpha n + 1/2.$$

Proof. Let $k = k_n$ as usual. Using Theorem 5.1, as well as Eqs. (7) and (21) gives

$$\begin{aligned} 0 \leqslant c_{n,k} - c_{n,k-1} &= M_k - M_{k-1} + \frac{n-3k+1}{2}\binom{n-k}{1} \\ &\leqslant (4\alpha - 1)\frac{k^2}{2} + \frac{(3k-n-1)(k-n)}{2}. \end{aligned}$$

Let

$$f(x) = (4\alpha - 1)x^2 + (3x - n - 1)(x - n) = (4\alpha + 2)x^2 - (4n+1)x + \left(n^2 + n\right).$$

The vertex of this parabola is at $x_0 = (4n+1)/(8\alpha + 4)$ and from Proposition 2.7 we have $k < n/2 < x_0$. Combining this with the fact that $f(k) \geqslant 0$ shows that k is at most the smaller of the two roots of $f(x)$ which is

$$r = \frac{4n + 1 - \sqrt{(4n+1)^2 - 4(n^2+n)(4\alpha+2)}}{8\alpha + 4}.$$

To complete the proof we need to show that $r \leqslant \alpha n + 1/2$. Rearranging terms in this last inequality and using (20) shows that we need to prove

$$\sqrt{(4n+1)^2 - 4(n^2+n)(4\alpha+2)} \geqslant 4\alpha n - 4\alpha - 1.$$

Since $n \geqslant 3$, the right-hand side of this last inequality is positive. So we can square it and use (20) again to reduce our task to proving $(16 - 40\alpha)n + (8\alpha - 8) \geqslant 0$. But this is true since $n \geqslant 3$ and the theorem is proved. □

6. Comments and conjectures

There are several ways in which this work could be continued. We list some of them here in the hopes that the reader will be interested.

1. We have already noted that the upper bound in Theorem 3.7 is not true for $\ell = 2$. However, numerical evidence indicates that the succeeding results are still valid, even though the proofs we have given will not work. In particular, we make the following conjecture.

Conjecture 6.1. *For $\ell = 2$ and $n > \ell$ we have*

$$M_n \leqslant \beta \frac{n^{\ell+1}}{(\ell+1)!} \quad \textit{and} \quad k_n \leqslant \alpha(n - \ell) + 1.$$

2. The lower bound given for k_n in Theorem 4.2 suffers from the fact that our demonstration only works for sufficiently large n. The most restrictive place where this is used is in the proof that inequality (19) holds and there we need n to be at least on the order of ℓ^3. But numerical calculations suggest that an even better bound holds for all n.

Conjecture 6.2. *For all $\ell \geqslant 2$ and $n > \ell$ we have*

$$k_n \geqslant \alpha(n-\ell).$$

We should note that there are examples where k_n is not the closest integer to αn. So, given the upper bound we have already proven, one can not hope to substantially improve upon this conjecture.

3. The reader will have noticed that the Continuity Theorem has been of fundamental importance in proving the results in this paper. This leads us to wonder if something can be said for a larger class of layered patterns q. By Theorem 2.1, one can still define k_n as the maximum length of the word remaining after removing the last layer of a q-optimal layered n-permutation. So we would like to be able to say something about the sequence k_n. There are some results in this regard in Price's thesis [5] for patterns with at most two layers and certain patterns with all layer lengths two.

Another of our main tools which might be amenable to generalization to other layered permutations is the Bimodal Theorem. One can still define $c_{n,i,q}$ to be the maximum number of copies of q in a layered n-permutation where the last layer has length $n-i$. Knowing the shape of the sequence $(c_{n,i,q})_{0\leqslant i<n}$ could be useful in getting information about the packing density of q.

4. Because of Theorem 1.1, it is easy to generalize the lower bound of Theorem 1.3 to all patterns. The proof is the same as the second proof of the lower bound in Theorem 3.8 and so is left to the reader.

Theorem 6.3. *If q is a pattern of length L and $n \geqslant L$ then*

$$M_{n,q} \geqslant \delta(q)\frac{(n-L+1)^L}{L!}.$$

We conjecture that the corresponding upper bound holds as well.

Conjecture 6.4. *If q is a pattern of length L and $n \geqslant L$ then*

$$M_{n,q} \leqslant \delta(q)\frac{n^L}{L!}.$$

5. Finally, we should point out that since Herb Wilf first defined packing densities in 1992 at the SIAM meeting on Discrete Mathematics, only packing densities of layered permutations (or permutations equivalent to layered permutations under one of the 8

routine symmetries) have been computed. The first open cases are of length four, where Albert, Atkinson, Handley, Holton, and Stromquist [1] gave the bounds

$$0.19657 \leqslant \delta(1342) \leqslant 2/9$$

and

$$51/511 \leqslant \delta(2413) \leqslant 2/9.$$

While we are hopeful that the approach presented in this paper (and in particular, generalizations of the Continuity and Bimodal Theorems) may prove fruitful in other layered cases, our approach seems to offer no additional hope in the nonlayered cases.

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