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An Analog of Schensted's Algorithm for Shifted Young Tableaux

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Schensted [*Canad. J. Math. 13* (1961)] constructed an algorithm giving a bijective correspondence between permutations and pairs of Young Tableaux. The author develops an analogous algorithm relating permutations and triples consisting of two Shifted Tableaux and a set. Various properties of this algorithm are also examined.

1. DEFINITIONS AND MOTIVATION

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a *partition* of the integer n if $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$ and $\sum_i \lambda_i = n$. A *generalized Young Tableau* (abbreviated *YT*) of shape λ is an array of n integers into r left-justified rows with λ_i integers in row i, such that the rows are non-decreasing and the columns are strictly increasing. For example $\begin{smallmatrix}1&1&2&3\\2&2&3&\\3&&&\end{smallmatrix}$ is a YT of shape (4, 3, 1). The YT is said to be *standard* (an SYT) if the integers in the tableau are 1, 2,..., n as in $\begin{smallmatrix}1&2&5\\3&4&\\6&&\end{smallmatrix}$.

A partition $\mu = (\mu_1, \mu_2, \ldots, \mu_t)$ is *strict* if $\mu_1 > \mu_2 > \cdots > \mu_t > 0$. A *generalized Shifted Young Tableau* (abbreviated ST) of shape μ is an array of n-integers into t rows with row i containing μ_i integers and indented $i - 1$ spaces, such that the rows are non-decreasing and the columns are strictly increasing. The ST is said to be *standard* (an SST) if the integers in the tableau are 1, 2,..., n. The *diagonal* of an ST is the set of first elements in each row. All other elements of the ST are *off-diagonal*. In the SST $\begin{smallmatrix}1&2&4\\&3&5\end{smallmatrix}$ the diagonal is $\{1, 3\}$.

Schensted's algorithm [3] establishes a one-to-one correspondence between permutations of the integers 1, 2,..., n and pairs of SYT of the same shape having n entries. For this bijection we write $\pi \leftrightarrow (P(\pi), Q(\pi))$ where $P(\pi)$ and $Q(\pi)$ are called the P and Q shapes of π respectively. An immediate corollary of this correspondence is the formula

$$\sum_\lambda (f_\lambda)^2 = n! \tag{1.1}$$

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where the sum is over all partitions λ of n, and f_λ is the number of SYT of shape λ. It follows from the work of Schur [4] on projective representations of the symmetric group that

$$\sum_{\mu} 2^{n-t}(g_\mu)^2 = n! \tag{1.2}$$

where the sum is over all strict partitions μ of n, g_μ is the number of SST of shape μ, and t is the number of parts of μ. Hence, there should be an algorithmic way to prove the following result.

THEOREM. *There exists a bijective correspondence between permutations of* 1, 2,..., n *and triples denoted* $\pi \leftrightarrow (P^*(\pi), Q^*(\pi), S^*(\pi))$ *where* $P^*(\pi)$, $Q^*(\pi)$ *are SST of the same shape having n entries, and where* $S^*(\pi)$ *is a subset of the off-diagonal elements of* $Q^*(\pi)$.

The construction of this algorithm provides a combinatorial proof of Eq. (1.2).

2. Algorithmic Proof of the Theorem

We first describe a procedure for inserting a positive integer, x, into an ST, Y^*, of shape $(\mu_1, \mu_2, \ldots, \mu_t)$. We emulate Knuth's presentation of Schensted's algorithm [2, Section 2] to facilitate comparison between the two processes. Imbed Y^* in an infinite array

$$\begin{array}{llll} y_{11}y_{12}y_{13}y_{14} & \cdots \\ \phantom{y_{11}}y_{22}y_{23}y_{24} & \cdots \\ \phantom{y_{11}y_{12}}y_{33}y_{34} & \cdots \\ \phantom{y_{11}y_{12}y_{13}}y_{44} & \cdots \end{array}$$

where

$$\begin{aligned} y_{ij} &= \text{the corresponding element of } Y^* \text{ if } i \leqslant t, j \leqslant \mu_i + i - 1 \\ &= \infty \text{ if } i > t \text{ or } j > \mu_i + i - 1 \end{aligned}$$

Thus, for all $j \geqslant i \geqslant 1$ we have

$$y_{ij} \leqslant y_{i(j+1)} \qquad \text{and} \qquad y_{ij} < y_{(i+1)j} \tag{2.1}$$

with the convention that $\infty < \infty$. In the following insertion procedure parenthesized statements serve to verify that conditions (2.1) still hold before each step; hence the tableau remains an ST. The occurrence of y_{ab} where $a > b$ in any parenthesized statement can be safely ignored.

INSERT* (x):

I*1. Set $k \leftarrow 1$, $x_1 \leftarrow x$, $d \leftarrow 0$

I*2. Set $i \leftarrow 1$, and set j to some value such that $y_{ij} = \infty$.

I*3. (Now $y_{(i-1)j} < x_k < y_{ij}$ and $x_k \neq \infty$). If $y_{i(j-1)}$ is defined and $x_k < y_{i(j-1)}$ decrease j by 1 and repeat this step. Otherwise, continue.

I*4. If $i = j$ and $d = 1$, return to step I*2. Otherwise, set $x_{k+1} \leftarrow y_{ij}$.

I*5. (Now $y_{i(j-1)} \leqslant x_k < x_{k+1} = y_{ij} \leqslant y_{i(j+1)}$, $y_{(i-1)j} < x_k < x_{k+1} = y_{ij} < y_{(i+1)j}$ and $x_k \neq \infty$) Set $y_{ij} \leftarrow x_k$

I*6. (Now $y_{i(j-1)} \leqslant y_{ij} = x_k < x_{k+1} \leqslant y_{i(j+1)}$, $y_{(i-1)j} < y_{ij} = x_k < x_{k+1} < y_{(i+1)j}$ and $x_k \neq \infty$) If $x_{k+1} \neq \infty$ and $i \neq j$ increase both i and k by 1 and return to I*3. If $x_{k+1} \neq \infty$ and $i = j$ increase k by 1, set $d \leftarrow 1$, and return to I*2. Otherwise, continue.

I*7 Set $s \leftarrow i$, $t \leftarrow j$ and terminate (Now $y_{st} \neq \infty$, $x_{k+1} = y_{s(t+1)} = y_{(s+1)j} = \infty$).

This algorithm computes

(i) a sequence of positive integers

$$x = x_1 < x_2 < \cdots < x_l < \infty \tag{2.2}$$

where x_k "bumps" x_{k+1} into the next lower row or, in certain cases described below, into the first row.

(ii) the coordinates (s, t) of x_l

(iii) a number $d = 0$ or 1 which counts the number of times a non-infinite diagonal element is displaced by an x_k. Note that in the case $d = 0$, this algorithm is exactly like Knuth's INSERT(x) in [2]. However, once an x_k displaces some $y_{ii} \neq \infty$, then $x_{k+1} (= y_{ii})$ is inserted into the first row of Y^*, bumping x_{k+2} into the second row, etc. But now we have the restriction in step I*4 that no $x_{k+m} (m > 0)$ can ever displace another diagonal element, infinite or non-infinite. If for some m, x_{k+m} is less than every element in row m then it is inserted in row 1. As an example, inserting $x = 2$ into $\begin{smallmatrix}135\\46\\7\end{smallmatrix}$ gives the array $\begin{smallmatrix}1246\\35\\7\end{smallmatrix}$ and the sequence $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 5$, $x_5 = 6$.

The deletion algorithm with s, t, d given; $d = 0$ or 1; and where $s = t$ forces $d = 0$ is

DELETE* (s, t, d):

D*1. Set $i \leftarrow s, j \leftarrow t, k \leftarrow 0, z_0 \leftarrow \infty$

D*2. (Now $y_{ij} < z_k < y_{(i+1)j}$ and $y_{ij} \neq \infty$) If $y_{i(j+1)} < z_k$ and $y_{i(j+1)} \neq \infty$ increase j by 1 and repeat this step. Otherwise set $z_{k+1} \leftarrow y_{ij}$ and continue.

D*3 (Now $y_{i(j-1)} \leqslant y_{ij} = z_{k+1} < z_k \leqslant y_{i(j+1)}$, $y_{(i-1)j} < y_{ij} = z_{k+1} < z_k < y_{(i+1)j}$ and $z_{k+1} \neq \infty$) Set $y_{ij} \leftarrow z_k$ and if $i = j$ set $d \leftarrow 0$

D*4. (Now $y_{i(j-1)} \leqslant z_{k+1} < z_k = y_{ij} \leqslant y_{i(j+1)}$, $y_{(i-1)j} < z_{k+1} < z_k = y_{ij} < y_{(i+1)j}$ and $z_{k+1} \neq \infty$) If $i \neq 1$ decrease i by 1, increase k by 1, and return to D*2. Otherwise continue.

D*5. (Now $i = 1$) If $d = 0$ go to D*8. Otherwise continue.

D*6. (Now $d = 1$) Increase k by 1, set r to some value such that $y_{(r+1)(r+1)} = \infty$.

D*7. If $y_{rr} > z_k$ reduce r by 1 and repeat this step. Otherwise set $i \leftarrow r, j \leftarrow r$ and go to D*2.

D*8. Set $x \leftarrow z_{k+1}$ and terminate (now $x \neq \infty$).

The parenthesized statements again show that at each step the tableau remains an ST. Note that the deletion process computes a sequence of positive integers

$$\infty > z_1 > z_2 > \cdots > z_l = x \tag{2.3}$$

where at most one of the z_k is a diagonal element. In fact (2.3) is just (2.2) written backwards as can be verified by applying DELETE*(1, 4, 1) to the example above.

To prove this in general, the case where $d = 0$ is trivial since then DELETE* reduces to Knuth's inverse DELETE. If $d = 1$, the same reasoning holds until some $z_k (= x_{l-k+1})$ is displaced from the top row. Now z_k is inserted in the lowest row containing elements $< z_k$. Indeed, x_{l-k+1} must have come from this row, since an element of the x sequence is only inserted in row 1 when it is less than every element in the row below it. Hence, we must have $x_{l-k} = z_{k+1}$. Finally, the last time a z_k returns to a lower row is when it displaces the diagonal element of that row which corresponds to the diagonal element displaced by the x sequence. Similar considerations show that if we compute the sequence (2.3) first and then perform INSERT*(x) with $x = z_l$ then we have $x_k = z_{l-k+1}$ for $1 \leqslant k \leqslant l$. This demonstrates that INSERT* and DELETE* are inverses of each other.

We are at last in a position to prove the theorem stated in section 1. Given a permutation of n $\pi = (a_1, a_2, \ldots, a_n)$ construct $P^*(\pi)$, $Q^*(\pi)$ and $S^*(\pi)$ as follows.

(i) Let $P_0^*(\pi) = Q_0^*(\pi) =$ the empty tableau $y_{ij} = \infty$ for all $j \geqslant i > 0$, and let $S_0^*(\pi) = \phi$

(ii) For $i = 1, 2, \ldots, n$ perform INSERT* (a_i) on $P_{i-1}^*(\pi)$ to determine $P_i^*(\pi)$, s_i, t_i, and d_i. Let $Q_i^*(\pi)$ be $Q_{i-1}^*(\pi)$ with i placed in position (s_i, t_i).

Let

$$\begin{aligned} S_i^*(\pi) &= S_{i-1}^* && \text{if} \quad d_i = 0 \\ &= S_{i-1}^* \cup \{i\} && \text{if} \quad d_i = 1. \end{aligned}$$

(iii) Let $P^*(\pi) = P_n^*(\pi)$, $Q^*(\pi) = Q_n^*(\pi)$, $S^*(\pi) = S_n^*(\pi)$

Clearly $P^*(\pi)$ is an SST. $Q^*(\pi)$ is also an SST because at every step we are placing i, which is larger than any element in $Q_{i-1}^*(\pi)$, at the end of a row and column. Finally $S^*(\pi)$ is a subset of the off-diagonal elements of $Q^*(\pi)$ because $i \in S^*(\pi)$ if and only if a non-infinite diagonal element is displaced during INSERT*(a_i). But after this displacement, no other diagonal elements (including ∞) can be displaced. Hence $s_i \neq t_i$.

Now given SST's P^* and Q^* of the same shape having n entries and given S^* a subset of the off-diagonal elements of Q^* we construct a permutation π of n by

(i) Let $P_n^* = P^*$, $Q_n^* = Q^*$, $S_n^* = S^*$

(ii) For $i = n, n-1, \ldots, 1$ find i in Q_i^* with coordinates (s_i, t_i) and let

$$\begin{aligned} d_i &= 0 && \text{if} \quad i \notin S_i^* \\ &= 1 && \text{if} \quad i \in S_i^*. \end{aligned}$$

Perform DELETE* (s_i, t_i, d_i) on P_i^* to determine P_{i-1}^* and a_i (the last member of sequence (2.3)). Let Q_{i-1}^* be Q_i^* with i removed, let

$$\begin{aligned} S_i^* &= S_i^* && \text{if} \quad d_i = 0 \\ &= S_i^* - \{i\} && \text{if} \quad d_i = 1. \end{aligned}$$

(iii) Let $\pi = (a_1, a_2, \ldots, a_n)$

This is a step by step reversal of the algorithm for building $P^*(\pi)$, $Q^*(\pi)$ and $S^*(\pi)$ hence the proof of the theorem is complete. The reader may find it instructive to try an example e.g. show that

$$(6, 7, 1, 4, 3, 5, 2) \leftrightarrow \left(\begin{matrix} 1246 \\ 35 \\ 7 \end{matrix} \quad \begin{matrix} 1237 \\ 45 \\ 6 \end{matrix} \quad \{3, 5, 7\} \right).$$

3. Properties and Characterizations

In general the properties of Schensted's correspondence do not seem to carry over to this new setting. However, it seems reasonable to expect the class of permutations such that $S^*(\pi) = \phi$ to behave in a "Schensted-like" manner. To make this precise we consider shiftable tableau.

Definition. A YT of shape λ, P, is *shiftable* if the tableau, P^*, gotten by indenting row i by $i - 1$ spaces is an ST. In other words, P is shiftable if λ is strict and each element of P is greater than its neighbor diagonally upward and to the right. In this case P^* is called the *shift* of P.

We also need a lemma.

Lemma. *Let $\pi = (a_1, a_2, \ldots, a_n)$ be a permutation, then the following are equivalent*

(i) $S^*(\pi) = \phi$

(ii) *The tableaux $P(\pi)$ and $Q(\pi)$ of Schensted's algorithm are shiftable and their shifts are $P^*(\pi)$ and $Q^*(\pi)$ respectively*

(iii) *The tableaux $P(\pi)$ and $Q(\pi)$ are shiftable*

Proof. (i) $\to$ (ii). We will show that in fact the tableaux $P_i(\pi)$, $Q_i(\pi)$ at the ith stage of Schensted's algorithm are shiftable with shifts $P_i^*(\pi)$, $Q_i^*(\pi)$ respectively. Induct on i. The case where $i = 1$ is trivial. Assume that $P_{i-1}^*(\pi)$, $Q_{i-1}^*(\pi)$ are the shifts of $P_{i-1}(\pi)$, $Q_{i-1}(\pi)$. But inserting a_i into $P_{i-1}(\pi)$ causes the same elements to be moved to the same relative positions as in the insertion of a_i into $P_{i-1}^*(\pi)$ (since $d_i = 0$). Hence, the statement holds after the ith insertion.

(ii) $\to$ (iii). Obvious

(iii) $\to$ (i). Let P^* and Q^* be the shifts of $P(\pi)$ and $Q(\pi)$ respectively. Then, by the theorem of section 1 there exists a unique permutation σ such that $P^*(\sigma) = P^*$, $Q^*(\sigma) = Q^*$, and $S^*(\sigma) = \phi$. Using the proof that (i) $\to$(ii) we have $P(\sigma) = P(\pi)$ and $Q(\sigma) = Q(\pi)$. Hence, $\sigma = \pi$ by the injectivity of Schensted's algorithm.

Many of the known results concerning Schensted's algorithm can now be seen to have analogs in the case where $S^*(\pi) = \phi$. A sampling of these is given in Proposition 1. The number following each assertion is the reference where the original statement and definitions can be found.

Proposition 1. *If $S^*(\pi) = \phi$ then*

(i) *The number of elements in the first k rows of $P^*(\pi)$ is the maximum length of a k-increasing subsequence of π.* [1]

(ii) *The elements of the ith row of* $P^*(\pi)$ *are precisely the 2nd coordinates of the set of source verticies in the ith inversion digraph of* π. *Similarly for* $Q^*(\pi)$ *and the* 1*st coordinates.* [2]

(iii) $S^*(\pi^{-1}) = \phi$ *and* $P^*(\pi^{-1}) = Q^*(\pi)$, $Q^*(\pi^{-1}) = P^*(\pi)$. [2]

All these properties follow directly from our lemma and the corresponding statements for Schensted's correspondence. It would also be nice to characterize those π with $S^*(\pi) = \phi$ in terms of the elements of π itself. To this end we define

DEFINITION. Permutations π and σ *differ by an adjacent transposition* if $\pi = (a_1, a_2 \cdots a_k, a_{k+1} \cdots a_n)$ and $\sigma = (a_1, a_2 \cdots a_{k+1}, a_k \cdots a_n)$. Let p be any property of permutations. We say that π and σ are *p-equivalent*, written $\pi \equiv \sigma\,(p)$, if there exists a sequence of permutations $\pi = \pi_1, \pi_2, \ldots, \pi_n = \sigma$ such that for all i

(i) π_i has property p

(ii) π_i and π_{i+1} differ by an adjacent transposition.

Note that $\equiv(p)$ is an equivalence relation.

PROPOSITION 2. *Let* p_1 *be the property* "$S^*(\pi) = \phi$" *then* $S^*(\pi) = S^*(\sigma) = \phi$ *implies* $\pi \equiv \sigma(p_1)$.

Proof. It suffices to prove that $\pi \equiv (1, 2, \ldots, n)(p_1)$ for any $\pi = (a_1, a_2, \ldots, a_n)$ with $S^*(\pi) = \phi$. If $a_n = n$ we are done by induction, so assume $a_n \neq n$. But if $S^*(\pi) = \phi$ then $a_n \neq 1$ (for $n > 1$).
Now for all $j \neq 1$

$$\begin{aligned}(1, 2, \ldots, n) &\equiv (1, 2 \cdots j-1, j+1, j \cdots n)(p_1)\\ &\equiv (1, 2 \cdots j+1, j+2, j \cdots n)(p_1)\\ &\;\vdots\\ &\equiv (1, 2 \cdots \hat{j} \cdots n, j)(p_1)\end{aligned}$$

where $\hat{j}$ indicates that j is deleted. Hence, by induction on n

$$\begin{aligned}(a_1, a_2, \ldots, a_n) &\equiv (1, 2 \cdots \hat{a}_n \cdots n, a_n)(p_1)\\ &\equiv (1, 2, \ldots, n)(p_1)\end{aligned}$$

As an example of what has been discussed so far, consider $\pi = (1, 3, 5, 2, 4)$ Then $P(\pi) = \begin{smallmatrix}1&2&4\\3&5\end{smallmatrix}$ and $Q(\pi) = \begin{smallmatrix}1&2&3\\4&5\end{smallmatrix}$ are both shiftable so $P^*(\pi) = \begin{smallmatrix}1&2&4\\&3&5\end{smallmatrix}$, $Q^*(\pi) = \begin{smallmatrix}1&2&3\\&4&5\end{smallmatrix}$ and $S^*(\pi) = \phi$. Also $(1, 3, 5, 2, 4) \equiv (1, 3, 2, 5, 4) \equiv (1, 3, 2, 4, 5) \equiv (1, 2, 3, 4, 5)(p_1)$.

Another problem is to characterize all permutations having a fixed P^* and Q^* shape.

PROPOSITION 3. *If* $\pi = (a_1, a_2, \ldots, a_n)$ *then the following are equivalent*

(i) $P^*(\pi) = Q^*(\pi) = 12 \cdots n$

(ii) *For all* i, $a_i = \min\{a_1, a_2, \ldots, a_i\}$ *or* $a_i = \max\{a_1, a_2, \ldots, a_i\}$

Proof. Call (ii) property p_2. A moments reflection will show that if π satisfies p_2 then we must have $P^*(\pi) = Q^*(\pi) = 12 \cdots n$. To show that these are the only ones it suffices to prove that there are exactly 2^{n-1} such permutations, since the correspondence $\pi \leftrightarrow (12 \cdots n, 12 \cdots n, S^*(\pi))$ is bijective and $S^*(\pi) \subseteq \{2, 3, \ldots, n\}$. Now there are exactly 2 choices for a_n i.e. 1 or n, 2 choices for a_{n-1}, etc. but only one element left over for a_1 so we are done.

COROLLARY. $P^*(\pi) = P^*(\sigma) = 12 \cdots n$ *if and only if* $\pi \equiv \sigma(p_2)$.

Proof. The "if" direction follows immediately from Proposition 3. The other implication is an easy induction based on the fact that the last element of π must be 1 or n. It is left to the reader.

We might hope that if p_3 was the property "$P^*(\pi) = P^*$, $Q^*(\pi) = Q^*$ with P^*, Q^* fixed" then we would have $\pi \equiv \sigma\ (p_3)$ for any π and σ with $P^*(\pi) = P^* = P^*(\sigma)$ and $Q^*(\pi) = Q^* = Q^*(\sigma)$. This is not true, however, as can be seen by taking $\pi = (1, 3, 2, 4)$ and $\sigma = (1, 4, 3, 2)$.

4. PROBLEMS AND CONJECTURE

There are still many unanswered questions in relation to this new algorithm and its properties. Some of these are listed here in the hope that the reader may ponder them.

(i) Schensted's algorithm can be generalized to give a bijective correspondence between matrices, A, with non-negative integer entries and pairs of generalized YT. Such a matrix is viewed as a lexicographically ordered two-row array where column $\genfrac{}{}{0pt}{}{i}{j}$ appears A_{ij} times. Then the elements of the bottom line are inserted into the P shape and those of the top line fill up the Q shape. The algorithm in section 2 can also be applied to such two-row arrays, the elements of S^* becoming coordinates of off-diagonal positions in Q^*. But the tableau Q^* may not turn out to be a generalized ST, as two equal integers may appear in the same column. The problem is to find a suitable extension of this algorithm to matrices.

(ii) Give a characterization of all permutations giving a fixed P^* and Q^* shape. Or characterize all permutations giving a fixed P^* shape.

CONJECTURE. *If p_4 is the property "$P^*(\pi) = P^*$ fixed" then $P^*(\pi) = P^*(\sigma) = P^*$ implies $\pi \equiv \sigma\ (p_4)$.* (This is the analog of a theorem of Knuth [2] but his proof does not work.)

(iii) What if we consider "shifted" tableaux where the rows are indented 2 spaces each? Is there still an algorithmic bijective correspondence? What about the theory of n-indented tableaux, or ones with arbitrary left-hand boundary?

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