Arithmetic Properties of Generalized Euler Numbers

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Abstract

The generalized Euler number $E_{n|k}$ counts the number of permutations of $\{1, 2, \ldots, n\}$ which have a descent in position m if and only if m is divisible by k. The classical Euler numbers are the special case when $k = 2$. In this paper, we study divisibility properties of a q-analog of $E_{n|k}$. In particular, we generalize two theorems of Andrews and Gessel [3] about factors of the q-tangent numbers.

1 Introduction

Let $\mathfrak{S}_n$ denote the *symmetric group* of all permutations of the set $\mathbf{n} = \{1, 2, \ldots, n\}$. The nth *Euler number*, E_n, can be defined as the number of permutations $\pi = a_1 a_2 \ldots a_n$ in $\mathfrak{S}_n$ that alternate, i.e.,

$$a_1 < a_2 > a_3 < \ldots$$

These numbers have a long and venerable history going back at least to André [1,2]. Comtet's book [4, p. 48] lists some of the classical properties of the E_n. In particular, the Euler numbers have exponential generating function

$$\sum_{n \geq 0} E_n x^n / n! = \tan x + \sec x.$$

For this reason the E_{2n+1} are called *tangent numbers* and the E_{2n} *secant numbers.*

The *descent set* of any $\pi = a_1 a_2 \ldots a_n$ is the set of indices

$$\mathrm{Des}(\pi) = \{i \ : \ 1 \leq i < n \text{ and } a_i > a_{i+1}\}.$$

Furthermore, the *generalized Euler*, $E_{n|k}$, counts the number of $\pi \in \mathfrak{S}_n$ such that $\mathrm{Des}(\pi) = \{k, 2k, 3k, \ldots\}$. We will also use $E_{\mathbf{n}|k}$ to denote the set of all such permutations. Clearly, $E_{n|2} = E_n$. As an example, we have

$$E_{\mathbf{5}|3} = \{12435,\ 13425,\ 23415,\ 12534,\ 13524,\ 14523,\ 23514,\ 24513,\ 34512\}.$$

We will be concerned with a certain q-analog of the generalized Euler numbers defined as follows. An *inversion* of $\pi = a_1 a_2 \ldots a_n$ is an out-of-order pair, namely (a_i, a_j) with $i < j$ and $a_i > a_j$. We let $\mathrm{inv}\,\pi$ denote the number of inversions of π. Following Stanley [7, pp. 147–9], define

$$E_{n|k}(q) = \sum_{\pi} q^{\mathrm{inv}(\pi)}. \tag{1}$$

Continuing our example from the previous paragraph,

$$\begin{aligned} E_{5|3}(q) &= q + q^2 + q^3 + q^2 + q^3 + q^4 + q^4 + q^5 + q^6 \\ &= q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6. \end{aligned}$$

It is well-known that the tangent numbers are divisible by high powers of 2. In [3], Andrews and Gessel show that both $(1+q)(1+q^2)\cdots(1+q^n)$ and $(1+q)^n$ divide the q-tangent number $E_{2n+1}(q)$. Our main theorem is a generalization of this result to $E_{n|k}(q)$. Let

$$[k] = [k]_q = 1 + q + q^2 + \cdots + q^{k-1}.$$

So $[k]_{q^i} = 1 + q^i + q^{2i} + \cdots + q^{(k-1)i}$.

Theorem 1.1 *Let k be prime and $1 \leq i \leq k-1$. Then*

1. $[k][k]_{q^2}[k]_{q^3}\cdots[k]_{q^n} \mid E_{nk+i|k}(q)$;
2. $[k]^n \mid E_{nk+i|k}(q)$.

The next section is devoted to proving a recursion and two lemmas that we will need for the proof of the previous theorem. Section 3 is devoted to the demonstration of the theorem itself. Finally, we close with a section of comments and open questions.

2 Lemmas

It will be useful to have a recursion relation for the $E_{n|k}(q)$. To state it we will need the *Gaussian polynomials* or *q-binomial coefficients*

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$$

where $[n]! = [n][n-1]\cdots[1]$. We assume $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ if $k > n$. It is well-known that these polynomials can be written as

$$\begin{bmatrix} n \\ k \end{bmatrix} = \sum_{\pi \in \mathfrak{S}_{n,k}} q^{\operatorname{inv} \pi}$$

where $\mathfrak{S}_{n,k}$ is the set of all permutations of k zeros and $n-k$ ones. Finally, let $\chi(P)$ be the *characteristic function* which is 1 if the statement P is true and 0 if it is false.

Proposition 2.1 *For $n \geq 0$ the $E_{n|k}(q)$ satisfies the following recursion*

$$E_{(n+1)|k}(q) = \sum_{m=1}^{\lfloor n/k \rfloor} \begin{bmatrix} n \\ mk-1 \end{bmatrix} q^{n-mk+1} E_{(mk-1)|k}(q) E_{(n-mk+1)|k}(q) + \chi(k \not| \, n) E_{n|k}(q)$$

with boundary condition $E_{0,k}(q) = 1$.

Proof. The initial condition is trivial. For the recurrence relation, consider all the indices i where one could have $a_i = n+1$ in some $\pi = a_1 a_2 \ldots a_{n+1} \in \mathfrak{S}_{n+1}$. Clearly, $i = n+1$ can occur iff $k \not| \, n$ and in this case all π ending in $n+1$ contribute $E_{n|k}(q)$ to the sum (1) for $E_{n+1|k}$.

The other possible positions for $n+1$ are at $i = mk$ for some m, $1 \leq m \leq \lfloor n/k \rfloor$. In this case, the inversions caused by $n+1$ are accounted for by q^{n-km+1}. The inversions of the elements $a_1 \dots a_{mk-1}$ and $a_{mk+1} \dots a_{n+1}$ among themselves are counted by $E_{(mk-1)|k}(q)$ and $E_{(n-mk+1)|k}(q)$, respectively. Finally, $\left[\begin{smallmatrix} n \\ mk-1 \end{smallmatrix}\right]$ takes care of inversions between these two sets of integers. ■

Given the form of the recursion in the previous proposition, it should come as no surprise that we will need two lemmas about divisibility properties of q-binomial coefficients .

Lemma 2.2 *Let k be a prime and suppose $0 \le i \le k-2$. Then for any non-negative integers n and m, we have the divisibility relation*

$$[k] \mid \begin{bmatrix} nk+i \\ mk-1 \end{bmatrix}.$$

Proof. For all non-negative integers n and m, we have

$$\begin{bmatrix} nk+i \\ mk-1 \end{bmatrix} = \frac{[nk+i][nk+i-1]\cdots[nk-mk+i+2]}{[mk-1]!}. \tag{2}$$

Since k is prime, all roots of $[k]$ are primitive kth roots of unity. If ζ is such a root then ζ is a root of $[l]$ iff $k|l$ and in that case it has multiplicity one. Thus the multiplicity of ζ as a root of the numerator of (2) is $n-(n-m)=m$ while in the denominator it is $m-1$. Thus $[k]$ divides the q-binomial coefficient as claimed. ■

Lemma 2.3 *Let k be prime and suppose $0 \le i \le k-2$. Then for any non-negative integers n and m, the expression*

$$\begin{bmatrix} nk+i \\ mk-1 \end{bmatrix} \frac{[k]_{q^{m-1}}[k]_{q^{m-2}}\cdots[k]}{[k]_{q^n}[k]_{q^{n-1}}\cdots[k]_{q^{n-m+1}}}$$

is a polynomial in q.

Proof. As with the previous lemma, we need only show that each root of unity which is a zero of the denominator appears with at least as large multiplicity in the numerator. We write the Gaussian polynomial as

$$\begin{bmatrix} nk+i \\ mk-1 \end{bmatrix} = \frac{(1-q^{nk+i})(1-q^{nk+i-1})\cdots(1-q^{nk-mk+i+2})}{(1-q^{mk-1})(1-q^{mk-2})\cdots(1-q^2)(1-q)} = P_1P_2 \tag{3}$$

where

$$P_1 = \frac{(1-q^{nk})(1-q^{(n-1)k})\cdots(1-q^{(n-m+1)k})}{(1-q^{(m-1)k})(1-q^{(m-2)k})\cdots(1-q^k)}$$

and P_2 contains all the other factors of $1-q^j$. Substituting $[k]_{q^j} = (1-q^{jk})/(1-q^j)$ into the expression in the statement of the lemma and doing some cancelation shows that

$$\begin{bmatrix} nk+i \\ mk-1 \end{bmatrix} \frac{[k]_{q^{m-1}}[k]_{q^{m-2}}\cdots[k]}{[k]_{q^n}[k]_{q^{n-1}}\cdots[k]_{q^{n-m+1}}} = P_3P_2 \tag{4}$$

where

$$P_3 = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q^{m-1})(1-q^{m-2})\cdots(1-q)}.$$

Let ζ be a primitive lth root of unity. Then since k is prime, either $\gcd(k,l)=1$ or $k|l$. In the former case, the multiplicities of ζ in the denominators of P_1P_2 and P_3P_2 are equal. Since the same is true of the numerators, P_3P_2 has no pole at ζ since P_1P_2 doesn't. If $k|l$ then ζ is neither a root nor a pole of P_2. Also $P_3 = (1-q^n)\left[\begin{smallmatrix} n-1 \\ m-1 \end{smallmatrix}\right]$ is a polynomial in q and so does not have ζ as a pole. Thus no primitive root of unity is a pole of P_2P_3 forcing it to be a polynomial. ■

3 Divisibility of the generalized q-Euler numbers

Now we are in the position to prove our main results.

Theorem 3.1 *Let k be prime and $1 \le i \le k-1$, then $E_{(nk+i)|k}(q)$ is divisible by $[k]^n$.*

Proof. We will induct on n. For $n=0$, the result is trivial. Suppose the result is true up to but not including n. First consider $i=1$. According to Proposition 2.1

$$E_{(nk+1)|k}(q) = \sum_{m=1}^{n} q^{nk-mk+1} \begin{bmatrix} nk \\ mk-1 \end{bmatrix} E_{(mk-1)|k}(q)E_{(nk-mk+1)|k}(q). \tag{5}$$

By induction $[k]^{m-1}$ and $[k]^{n-m}$ divide $E_{(mk-1)|k}(q)$ and $E_{(nk-mk+1)|k}(q)$, respectively. But by Lemma 2.2, $[k]$ is a factor of the corresponding q-binomial coefficient in (5). So $E_{(nk+1)|k}(q)$ is divisible by $[k]^{m-1+n-m+1} = [k]^n$ as desired. The case when $2 \le i \le k-1$ is similar, with the extra term from the recursion in Proposition 2.1 being taken care of by the case for $i-1$. ■

Theorem 3.2 *Let k be prime and $1 \le i \le k-1$, then $E_{(nk+i)|k}(q)$ is divisible by $[k][k]_{q^2}[k]_{q^3}\cdots[k]_{q^n}$.*

Proof. As before, we will induct on n with $n=0$ being trivial. Suppose the result is true up to but not including n. When $i=1$ we have (5) again and examine each of its terms. By the induction hypothesis

$$E_{(mk-1)|k}(q) = [k][k]_{q^2}[k]_{q^3}\cdots[k]_{q^{m-1}}Q_1$$

and

$$E_{(nk-mk+1)|k}(q) = [k][k]_{q^2}[k]_{q^3}\cdots[k]_{q^{n-m}}Q_2$$

where Q_1 and Q_2 are polynomials in q. Then

$$\begin{aligned} &\begin{bmatrix} nk \\ mk-1 \end{bmatrix} E_{(mk-1)|k}(q)E_{(nk-mk+1)|k}(q) \\ &\quad \begin{bmatrix} nk \\ mk-1 \end{bmatrix} \frac{[k]_{q^{m-1}}[k]_{q^{m-2}}\cdots[k]}{[k]_{q^n}[k]_{q^{n-1}}\cdots[k]_{q^{n-m+1}}}[k][k]_{q^2}[k]_{q^3}\cdots[k]_{q^n}Q_1Q_2. \end{aligned}$$

By Lemma 2.3, the q-binomial coefficient times the fraction is a polynomial in q. So $[k][k]_{q^2}[k]_{q^3}\cdots[k]_{q^n}$ is a factor of every term in (5) and thus of $E_{(nk+1)|k}(q)$. The case $2 \leq i \leq k-1$ is handled as in the proof of Theorem 3.1. ■

4 Comments and open questions

By setting $q = 1$ in either Theorem 3.1 or Theorem 3.2 we get the following corollary.

Corollary 4.1 *Let k be prime and $1 \leq i \leq k-1$, then $E_{(nk+i)|k}$ is divisible by k^n.*

It is well known that for $k = 2$ (the tangent numbers)

$$2^{2n} \mid (n+1)E_{2n+1} \tag{6}$$

and that the corresponding quotient, called a *Genocchi number*, is odd. Thus it is not surprising that better divisibility results can be obtained when $q = 1$ for general primes k. In particular, Gessel and Viennot [6] have shown that

$$k^{\lceil \frac{nk-j}{k-1} \rceil} \mid \binom{nk}{j} E_{(nk-j)|k}. \tag{7}$$

Note that this reduces to (6) when $k = 2$ and $j = 1$. This raises a couple of questions. Is it true that the associated quotient in (7) is relatively prime to k? Can these results be extended to the case of arbitrary q? With regards to the second query, the reader should consult Foata's article [5] which provides some answers in the case $k = 2$.

References

1. André, D.: Développements de $\sec x$ et de $\operatorname{tang} x$, *C. R. Acad. Sci. Paris*, **88**, 965–967 (1879).

2. André, D.: Sur les permutations alternées, *J. Math. Pures Appl.,* **7**, 167–184 (1881).

3. Andrews, G., Gessel, I.: Divisibility properties of the q-tangent numbers, *Proc. Amer. Math. Soc.*, **68**, 380–384 (1978).

4. Comtet, L., *Advanced Combinatorics,* D. Reidel Pub. Co., 1974.

5. Foata, D.: Further divisibility properties of the q-tangent numbers, *Proc. Amer. Math. Soc.*, **81**, 143–148 (1981).

6. Gessel, I., Viennot, G.: Generalized tangent and Genocchi numbers, preprint.

7. Stanley, R.P., *Enumerative Combinatorics, Volume I,* Wadsworth and Brooks/Cole, 1986.