Journal of Combinatorial Theory, Series A 148 (2017) 275–315

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series A

www.elsevier.com/locate/jcta

Antipodes and involutions

Carolina Benedetti, Bruce E. Sagan

Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA

A R T I C L E I N F O

Article history:
Received 1 November 2015

Keywords:
Acyclic orientation
Antipode
Involution
Malvenuto–Reutenauer Hopf algebra
$\mathfrak{m}$QSym
NSym
Poirier–Reutenauer Hopf algebra
QSym
Shuffle Hopf algebra
Takeuchi formula

A B S T R A C T

If H is a connected, graded Hopf algebra, then Takeuchi's formula can be used to compute its antipode. However, there is usually massive cancellation in the result. We show how sign-reversing involutions can sometimes be used to obtain cancellation-free formulas. We apply this idea to nine different examples. We rederive known formulas for the antipodes in the Hopf algebra of polynomials, the shuffle Hopf algebra, the Hopf algebra of quasisymmetric functions in both the monomial and fundamental bases, the Hopf algebra of multi-quasisymmetric functions in the fundamental basis, and the incidence Hopf algebra of graphs. We also find cancellation-free expressions for particular values of the antipode in the immaculate basis for the noncommutative symmetric functions as well as the Malvenuto–Reutenauer and Poirier–Reutenauer Hopf algebras, some of which are the first of their kind. We include various conjectures and suggestions for future research.

E-mail addresses: caro.benedetti@gmail.com (C. Benedetti), sagan@math.msu.edu (B.E. Sagan).

http://dx.doi.org/10.1016/j.jcta.2016.12.005

1. Introduction

Let $(H, m, u, \Delta, \epsilon)$ be a bialgebra over a field $\mathbb{F}$. Call H *graded* if it can be written as $H = \oplus_{n\geq 0} H_n$ so that

1. $H_i H_j \subseteq H_{i+j}$ for all $i, j \geq 0$,
2. $\Delta H_n \subseteq \oplus_{i+j=n} H_i \otimes H_j$ for all $n \geq 0$, and
3. $\epsilon H_n = 0$ for all $n \geq 1$.

If $H_0 \cong \mathbb{F}$, then we say that H is *connected*. Takeuchi [23] showed that if a bialgebra is graded and connected, then it is a Hopf algebra and gave an explicit formula for its antipode. To state his result, define a projection map $\pi : H \to H$ by linearly extending

$$\pi|_{H_n} = \begin{cases} 0 & \text{if } n = 0, \\ I & \text{if } n \geq 1, \end{cases} \tag{1}$$

where 0 and I are the zero and identity maps, respectively.

Theorem 1.1 *([23]). Let H be a connected graded bialgebra. Then H is a Hopf algebra with antipode*

$$S = \sum_{k\geq 0} (-1)^k m^{k-1} \pi^{\otimes k} \Delta^{k-1}, \tag{2}$$

where we let $m^{-1} = u$ and $\Delta^{-1} = \epsilon$. □

Equation (2) has the advantage of giving an explicit formula for the antipode. But it is usually not the most efficient way to calculate S as there is massive cancellation in the alternating sum. One of the standard combinatorial techniques for eliminating cancellations is the use of sign-reversing involutions. Let A be a set and $\iota : A \to A$ be an involution on A so that ι is composed of fixed points and two-cycles. Suppose that A is equipped with a sign function $\operatorname{sgn} : A \to \{+1, -1\}$. The involution ι is *sign reversing* if, for each two-cycle (a, b), we have $\operatorname{sgn} a = -\operatorname{sgn} b$. It follows that

$$\sum_{a\in A} \operatorname{sgn} a = \sum_{a\in F} \operatorname{sgn} a,$$

where F is the set of fixed points of ι. Furthermore, if all elements of F have the same sign, then

$$\sum_{a\in A} \operatorname{sgn} a = \pm|F|,$$

where the bars denote cardinality.

The purpose of the current work is to show how sign-reversing involutions can be used to give cancellation-free formulas for graded connected Hopf algebras. We give nine different examples of this technique. The first, in Section 2, is an application to the Hopf algebra of polynomials. Of course, it is easy to derive the formula for S in this case by other means. But the ideas of splitting and merging which will appear over and over again can be seen here in their simplest form. Next, we consider the shuffle Hopf algebra where, again, splitting and merging provide a simple proof. One also sees why applying S yields the reversed word as it appears naturally as the unique fixed point of our involution. More complicated applications appear in Sections 4 and 5 where we consider the Hopf algebra of quasisymmetric functions in the monomial and fundamental bases. Motivated by ideas from K-theory, Lam and Pylyavskyy [15] defined multi-analogues of several Hopf algebras. Very recently, Patrias [20] derived cancellation-free expressions for their antipodes and we show how our method can be used to obtain one of them in Section 6. Next we give an involution proof of a formula of Humpert and Martin [14] for the antipode in the incidence Hopf algebra on graphs. Again, the acyclic orientations which are counted by the coefficients appear naturally when finding the fixed points of the involution. We end with three examples involving the immaculate basis of the Hopf algebra of noncommutative symmetric functions defined by Berg et al. [5], the Malvenuto–Reutenauer Hopf algebra of permutations [17] and the Poirier–Reutenauer Hopf algebra of Young tableaux [21]. Some of these expressions are the first cancellation-free ones in the literature. Aguiar and Mahajan [1] provided a cancellation-free antipode formula for the Malvenuto–Reutenauer Hopf algebra using Hopf monoids. We recover some of their results using certain involutions, and appealing only to the Hopf algebra structure. We end with a section about future research and open problems, as well as noting other recent work where our technique has been applied.

We should mention that various other researchers have been studying cancellation-free formulae of antipodes. For example, Méndez and Liendo [18] have constructed a Hopf algebra associated with any symmetric set operad. They then give a combinatorial formula for its antipode using Schröder trees. In another direction, Menous and Patras [19] generalize the forest formula for computing the antipode of the Hopf algebras of Feynman diagrams in perturbative quantum field theory, showing that it can be used in arbitrary right-handed polynomial Hopf algebras.

2. The polynomial Hopf algebra

In this section we will use a sign-reversing involution to derive the well-known formula for the antipode in the polynomial Hopf algebra $\mathbb{F}[x]$. We need some combinatorial preliminaries. If n is a nonnegative integer, then let $[n] = \{1, 2, \ldots, n\}$. An *ordered set partition* of $[n]$ is a sequence of nonempty disjoint subsets $\pi = (B_1, B_2, \ldots, B_k)$ such that $\uplus_i B_i = [n]$ where $\uplus$ is disjoint union. We denote this relation by $\pi \models [n]$. The B_i are called *blocks* and since they are sets we are free to always list their elements in a canonical order which will be increasing. We will also usually leave out the curly brackets and the commas within each block, although we will retain the commas separating the blocks. So, for example $(13, 2)$ has blocks $B_1 = \{1, 3\}$ and $B_2 = \{2\}$; the partition $(2, 3, 1)$ has blocks $B_1 = \{2\}$, $B_2 = \{3\}$, $B_3 = \{1\}$; and (123) has a single block $\{1, 2, 3\}$. Finally, it will sometimes be convenient to allow some of the B_i to be empty, in which case we will write $\pi \models_0 [n]$.

Since the antipode is linear, it suffices to know its action on a basis. Here we use the standard basis for $\mathbb{F}[x]$.

Theorem 2.1. *In $\mathbb{F}[x]$ we have*

$$S(x^n) = (-1)^n x^n.$$

Proof. To apply Takeuchi's formula, we first need to describe $\Delta^{k-1}(x^n)$. By definition

$$\Delta(x) = 1 \otimes x + x \otimes 1 = \sum_{(B_1,B_2)\models_0[1]} x^{|B_1|} \otimes x^{|B_2|}.$$

It follows from coassociativity that

$$\Delta^{k-1}(x) = \sum_{(B_1,\dots,B_k)\models_0[1]} x^{|B_1|} \otimes \cdots \otimes x^{|B_k|},$$

and since Δ is an algebra map

$$\Delta^{k-1}(x^n) = (\Delta^{k-1}x)^n = \sum_{(B_1,\dots,B_k)\models_0[n]} x^{|B_1|} \otimes \cdots \otimes x^{|B_k|}.$$

Plugging this into equation (2) and remembering that π kills anything in H_0 gives

$$S(x^n) = \sum_{k\geq 0}(-1)^k \sum_{(B_1,\dots,B_k)\models[n]} x^{|B_1|}\dots x^{|B_k|}. \tag{3}$$

Since $|B_1| + \cdots + |B_k| = n$ whenever $(B_1, \dots, B_k) \models [n]$, the previous equation simplifies to

$$S(x^n) = x^n \sum_{k\geq 0} \sum_{(B_1,\dots,B_k)\models[n]} (-1)^k.$$

The last displayed equation shows that we will be done if we can find a sign-reversing involution ι on the set

$$A = \bigcup_{k\geq 0} \{\pi = (B_1, \dots, B_k)\ :\ \pi \models [n]\},$$

where

$$\mathrm{sgn}(B_1, \ldots, B_k) = (-1)^k,$$

and ι has a single fixed point which is

$$\phi = (n, n-1, \ldots, 1). \tag{4}$$

This involution will be built out of two other maps. If $\pi = (B_1, \ldots, B_k)$, then the result of *merging* blocks B_i and B_{i+1} is the ordered partition $\mu_i(\pi)$ obtained by replacing these two blocks by $B_i \cup B_{i+1}$. For example, if $\pi = (5, 3, 249, 16, 78)$, then $\mu_3(\pi) = (5, 3, 12469, 78)$. If $|B_i| \geq 2$, then the result of *splitting* $B_i = a_1 \ldots a_j$ (where, as usual, the elements of the block are listed in increasing order) is the ordered partition $\sigma_i(\pi)$ obtained by replacing B_i by the ordered pair $a_1, a_2 \ldots a_j$. Returning to our example, we have $\sigma_3(\pi) = (5, 3, 2, 49, 16, 78)$.

To define $\iota(\pi)$ where $\pi = (B_1, \ldots, B_k)$ we find the least index l, if any, such that either $|B_l| \geq 2$ or $B_l = \{a\}$ with $a < \min B_{l+1}$. If there is such an index, then we let

$$\iota(\pi) = \begin{cases} \sigma_l(\pi) & \text{if } |B_l| \geq 2, \\ \mu_l(\pi) & \text{else.} \end{cases}$$

Otherwise $\iota(\pi) = \pi$. Continuing with $\pi = (5, 3, 249, 16, 78)$ from the previous paragraph, we can not have $l = 1$ or 2 since $5 > 3$ and $3 > 2$. But $|B_3| \geq 2$ which results in $\iota(\pi) = \sigma_3(\pi) = (5, 3, 2, 49, 16, 78)$.

It is clear from the definition that ι is a sign-reversing map. To show that $\iota^2(\pi) = \pi$, it suffices to consider the case where π is not a fixed point. Given the definition of the index l, we must have $\pi = (a_1, a_2, \ldots, a_{l-1}, B_l, \ldots, B_k)$ where $a_1 > a_2 > \cdots > a_{l-1} > \min B_l$. Suppose first that $|B_l| \geq 2$ and let $B_l = a_l a_{l+1} \ldots a_m$ as well as $B'_l = a_{l+1} \ldots a_m$. Then we have

$$\iota(\pi) = \sigma_l(\pi) = (a_1, \ldots, a_l, B'_l, B_{l+1}, \ldots, B_k).$$

Furthermore,

$$\iota^2(\pi) = \mu_l(a_1, \ldots, a_l, B'_l, B_{l+1}, \ldots, B_k)) = \pi$$

because $a_1 > \cdots > a_l < a_{l+1} = \min B'_l$. The demonstration that $\iota^2(\pi) = \pi$ when $|B_l| = 1$ is similar.

There remains to show that the only fixed point of ι is ϕ as defined by equation (4). But if π is a fixed point, then the index l does not exist which implies $\pi = (a_1, a_2, \ldots, a_n)$ with $a_1 > a_2 > \cdots > a_n$. Clearly the only ordered partition of this type is $\pi = \phi$. □

3. The shuffle Hopf algebra

We next use the split-merge technique to derive the antipode of the shuffle Hopf algebra. Let A be a finite alphabet and consider the Kleene closure A^* of all words

$w = a_1 \dots a_n$ over A. We let $l(w) = n$ denote the *length* of w. The underlying vector space of the *shuffle Hopf algebra* is the set of formal sums $\mathbb{F}A^*$. The product is given by shuffling

$$v \cdot w = \sum_{u \in v \sqcup\!\sqcup w} u,$$

where $v \sqcup\!\sqcup w$ indicates all $\binom{l(v)+l(w)}{l(v)}$ interleavings of v and w. Note that if different interleavings result in the same final word, then they are considered distinct and so such words are counted with multiplicity, for example, $ab \cdot a = 2aab + aba$. The coproduct is

$$\Delta w = \sum_{uv=w} u \otimes v,$$

where uv denotes concatenation, not product, and we permit u or v to be empty. To state the formula for the antipode we will need, for $w = a_1 a_2 \dots a_n$, the *reversal operator* $\operatorname{rev} w = a_n a_{n-1} \dots a_1$.

Theorem 3.1. *The antipode in $\mathbb{F}A^*$ is given by*

$$S(w) = (-1)^{l(w)} \operatorname{rev} w.$$

Proof. Applying Takeuchi, we see that

$$S(w) = \sum_{k \geq 1} (-1)^k \sum_{w_1 \dots w_k = w} w_1 \cdot \ \dots \ \cdot w_k, \tag{5}$$

where none of the w_i are empty. Assume that $w = a_1 a_2 \dots a_n$ where the a_i are considered distinct variables. Once we have proved the result for such w, the general case will follow by specialization of the a_i. We will use similar reasoning in future proofs without comment. Because of the distinctness condition, $\operatorname{rev} w$ only occurs in the term $w_1 \cdot \dots \cdot w_n$ and does so with sign $(-1)^n$. So it suffices to give a sign-reversing involution on the rest of the words in the sum.

Let $v \neq \operatorname{rev} w$ be a word resulting as a shuffle in the term $w_1 \cdot \ \dots \ \cdot w_k$ of (5). Find the largest index $j \geq 0$ such that

1. $l(w_1) = \cdots = l(w_j) = 1$ (which implies $w_i = a_i$ for $i \leq j$), and
2. $a_j a_{j-1} \dots a_1$ is a subword of v.

Now a_{j+1} is the first letter of w_{j+1}. If a_j is to the left of a_{j+1} in v, then v will also be a shuffle of opposite sign in the merged product

$$a_1 \cdot a_2 \cdot \ \dots \ \cdot a_{j-1} \cdot a_j w_{j+1} \cdot w_{j+2} \cdot \ \dots \ \cdot w_k.$$

Our involution will pair these two copies of v. If a_j is to the right of a_{j+1} in v, then we must have $l(w_{j+1}) > 1$ because, if not, then either j would not have been maximum or $v = \operatorname{rev} w$. Thus v will also be a shuffle of opposite sign in the split product

$$a_1 \cdot \ a_2 \cdot \ \ldots \ \cdot a_j \cdot a_{j+1} \cdot w'_{j+1} \cdot w_{j+2} \cdot \cdots \cdot w_k,$$

where w'_{j+1} is w_{j+1} with a_{j+1} removed. It is easy to see that these two operations are inverses and so we are done. □

4. The monomial basis of QSym

Quasisymmetric functions were introduced by Gessel [11] to study properties of P-partitions where P is a poset. Malvenuto and Reutenauer [16] then showed that the vector space, QSym, of quasisymmetric functions can be given a Hopf algebra structure and that its dual is related to Solomon's decent algebra. We wish to use involutions to rederive known formulas for the antipode acting on two bases for QSym. We start with the monomial basis. The formula for S in this basis was derived independently by Ehrenborg [9], and by Malvenuto and Reutenauer [16]. Here there will turn out to be more than one term in the final sum even though it is cancellation free. But the split-merge method will show how these summands appear naturally as fixed points of the involution.

Let $\mathbf{x} = \{x_1, x_2, \ldots\}$ be a countably infinite set of variables. Vector space bases for QSym are indexed by compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ which are sequences of positive integers called parts. The number of parts of α is called its *length* and denoted $l(\alpha)$. The *monomial quasisymmetric function* corresponding to α is defined by

$$M_\alpha = M_\alpha(\mathbf{x}) = \sum_{i_1 < i_2 < \cdots < i_l} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_l}^{\alpha_l}.$$

The M_α form a basis for QSym. The product in QSym is the normal product of power series. The coproduct is given by

$$\Delta M_\alpha = \sum_{\beta\gamma=\alpha} M_\beta \otimes M_\gamma,$$

where $\beta\gamma$ is concatenation with the same conventions as in the shuffle algebra. Applying Takeuchi's formula, we obtain

$$S(M_\alpha) = \sum_{k\geq 1} (-1)^k \sum_{\alpha_1 \ldots \alpha_k = \alpha} M_{\alpha_1} \cdot M_{\alpha_2} \cdot \ \ldots \ \cdot M_{\alpha_k}, \qquad (6)$$

where all the α_i are nonempty. We refer to the terms corresponding to a given index k in the inner sum as the *kth summand* of $S(M_\alpha)$. We will use the notation

$$M_{\alpha_1} \cdot M_{\alpha_2} \cdot \ \ldots \ \cdot M_{\alpha_k} = \alpha_1 \cdot \alpha_2 \cdot \ \ldots \ \cdot \alpha_k. \tag{7}$$

Note that, again, "$\cdot$" is being used to distinguish multiplication from concatenation.

Suppose that α has length $l(\alpha) = l$. Our strategy will be to cancel all terms in the kth summand of (6) into terms from either the $(k-1)$st or $(k+1)$st summand for $k < l$. A term from the lth summand will either cancel with one from the $(l-1)$st summand or be a fixed point. We first need to characterize the terms which can occur in the product (7). To do this, we need to recall the notion of a quasishuffle. Let A be a set of variables and let v be a vector whose components are sums of the variables. Given $B \subseteq A$ then the *restriction of v to B* is the vector $v|_B$ obtained by setting the variables not in B equal to zero and eliminating any components which are completely zeroed out in this way. For example, if $v = (c+d+e, b+f, a+c)$ and $B = \{c, d\}$, then $v|_B = (c+d, c)$. Now consider compositions α and β as two sets of distinct variables. In this context, their *quasishuffle* is a vector v containing only these variables such that $v|_\alpha = \alpha$ and $v|_\beta = \beta$. We let

$$\alpha \,\overline{\sqcup\!\sqcup}\, \beta = \{v \ : \ v \text{ is a quasishuffle of } \alpha \text{ and } \beta\}.$$

For example

$$(a,b) \,\overline{\sqcup\!\sqcup}\, (c) = \{(a,b,c), (a,b+c), (a,c,b), (a+c,b), (c,a,b)\}.$$

It is well known that

$$\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_k = \sum_{v \in \alpha_1 \,\overline{\sqcup\!\sqcup}\, \alpha_2 \,\overline{\sqcup\!\sqcup}\, \ldots \,\overline{\sqcup\!\sqcup}\, \alpha_k} v.$$

To state the formula for the antipode, we need two more notions. If $\alpha = (a_1, a_2, \ldots, a_l)$ is a composition, then its *reversal* is the composition

$$\operatorname{rev} \alpha = (a_l, a_{l-1} \ldots, a_1),$$

just as for words. We will also use the refinement partial order on compositions. Define $\beta \geq \alpha$ to mean β is a coarsening of α, that is, the parts of β are obtained by adding together adjacent parts of α.

Theorem 4.1 *([9,16]). The antipode in the monomial basis of* QSym *is given by*

$$S(M_\alpha) = (-1)^{l(\alpha)} \sum_{\beta \geq \operatorname{rev}(\alpha)} M_\beta.$$

Proof. Let $l = l(\alpha)$. We first define the action of the splitting operator. It will be convenient to define σ on pairs (π, v) where v is a term in the product $\pi = \alpha_1 \cdot \ \ldots \ \cdot \alpha_k$ and $k < l$. Since $k < l$, there must be an index j with $l(\alpha_j) \geq 2$. Let j be the smallest such index, so that $l(\alpha_1) = \cdots = l(\alpha_{j-1}) = 1$. The splitting operator is then defined to be $\sigma(\pi, v) = (\pi', v)$ where

$$\pi' = \alpha_1 \cdot \ldots \cdot \alpha_{j-1} \cdot (d) \cdot \alpha'_j \cdot \alpha_{j+1} \cdot \ldots \cdot \alpha_k, \tag{8}$$

d is the first element of α_j, and α'_j is α_j with d removed. Note that this is well defined since if v is a term in π, then it must also be a term in π' because the variables in α'_j are a subset of the ones in α_j. For example, suppose $\pi = (a) \cdot (b) \cdot (c, d, e) \cdot (f, g)$ which contains the term $v = (c+f, d, a+b+e+g)$. In this case, $j = 3$ and $\pi' = (a) \cdot (b) \cdot (c) \cdot (d, e) \cdot (f, g)$. Note that σ can be applied to any pair (π, v) in a product with $k < l$.

The set of pairs to which we can apply the merge map μ is more restricted. For the rest of the proof, we assume that α has its component variables listed in lexicographic order. Consider a pair (π, v) where π has the form (7). In order for μ to be the inverse of σ, we can only merge α_j and α_{j+1} if $l(\alpha_1) = \cdots = l(\alpha_j) = 1$. Given a quasishuffle v in the product π, find the smallest index j, if any, satisfying

(i) $l(\alpha_1) = \cdots = l(\alpha_j) = 1$, and
(ii) if $B = \alpha_j \alpha_{j+1}$, then $v|_B = (d, \ldots, e)$ where the elements $d, \ldots, e$ are listed in increasing lexicographic order (as are the elements of α).

Finally, we define $\mu(\pi, v) = (\pi', v)$ where

$$\pi' = \alpha_1 \cdot \ldots \cdot \alpha_{j-1} \cdot \alpha_j \alpha_{j+1} \cdot \alpha_{j+2} \cdot \ldots \cdot \alpha_k.$$

Note that condition (ii) and the fact that v is a term in π imply v is also a term in π' and so the map is well defined. To illustrate, suppose that we have (π, v) where $\pi = (a) \cdot (b) \cdot (c) \cdot (d, e) \cdot (f, g)$ and $v = (c + f, d, a + b + e + g)$. We can not have $j = 1$ because then $B = \{a, b\}$ and $v|_B = (a + b)$ with a and b in the same component. Similarly $j = 2$ will not work since then $B = \{b, c\}$ and $v|_B = (c, b)$ which is not in lexicographic order. But $j = 3$ satisfies both conditions, giving $\mu(\pi, v) = (\pi', v)$ where $\pi' = (a) \cdot (b) \cdot (c, d, e) \cdot (f, g)$. So μ inverts the action of σ in the previous example.

To define the involution ι, take a pair (π, v) and define $\iota(\pi, v) = \mu(\pi, v)$ if an index satisfying (i) and (ii) above can be found. If there is no index, then there are two possibilities. One is that there is an index j such that (i) is true, but $l(\alpha_{j+1}) \geq 2$ and d, the sole element of α_j is in the same component as or to the right of the leftmost element of α_{j+1}. In this case we define $\iota(\pi, v) = \sigma(\pi, v)$. The minimality of j implies that ι as defined thus far is an involution, and it is clearly sign reversing.

The only other possibility is that there is no index j satisfying (i) and (ii) because $l(\alpha_i) = 1$ for all i and (ii) is never true. In this case we must have $k = l$, so that v is a quasishuffle from the last summand with sign $(-1)^l$. We claim that in this case we have that (ii) is not satisfied if and only if $v \geq \text{rev}(\alpha)$. Indeed $v \not\geq \text{rev}(\alpha)$ is equivalent to the existence of a pair of consecutive letters $B = \{d, e\}$ appearing in lexicographic order in v. Thus $v|_B = (d, e)$ which is the same as saying that (ii) will be satisfied for some index. So the fixed points of the involution give us exactly the quasishuffles we need for the M_β with $\beta \geq \text{rev}(\alpha)$. □

5. The fundamental basis for QSym

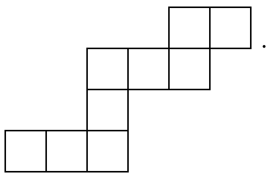
The *fundamental quasisymmetric function* corresponding to a partition α can be defined as

$$F_\alpha = \sum_{\beta \leq \alpha} M_\beta.$$

The formula for S in the fundamental basis first occurs in [16]. Our proof for the formula for the antipode of QSym in the fundamental basis will be very similar to the one for the monomial basis. We will arrange the notation and exposition to emphasize this fact.

Associated with any composition $\alpha = (a_1, \ldots, a_l)$ is its *rim-hook diagram* which has a_i cells in the ith row from the bottom and the last cell of row a_i is in the same column as the first cell of row a_{i+1}. We make no distinction between a composition and its diagram. For example

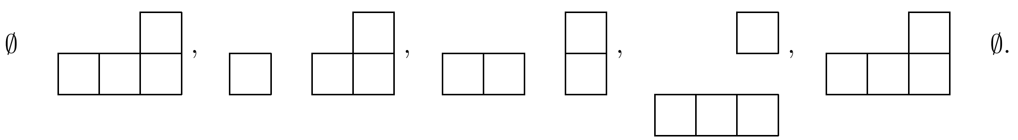
$\alpha = (3, 1, 3, 2) =$



A *cut-edge* of α is an edge which is the first vertical edge of α_1, the last vertical edge of α_l, or an edge bounding two cells of α. Separating α into pieces along a cut-edge results in two diagrams β to the southwest and γ to the northeast. In this case we write $\alpha = \beta|\gamma$. The coproduct applied to a fundamental quasisymmetric function is then

$$\Delta F_\alpha = \sum_{\beta|\gamma=\alpha} F_\beta \otimes F_\gamma.$$

To illustrate, if $\alpha = (3, 1)$ then the various pairs for β and γ are, as the cut-edge travels from southwest to northeast,



Thus

$$\Delta(F_{(3,1)}) = 1 \otimes F_{(3,1)} + F_{(1)} \otimes F_{(2,1)} + F_{(2)} \otimes F_{(1,1)} + F_{(3)} \otimes F_{(1)} + F_{(3,1)} \otimes 1.$$

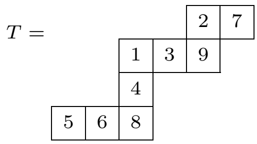


Fig. 1. An example for computing the row word.

To describe the product of fundamental quasisymmetric functions we associate compositions with permutations. Write $\alpha = (\alpha_1, \ldots, \alpha_l) \models n$ or $|\alpha| = n$ if $\sum_i \alpha_i = n$. There is a canonical bijection between $\alpha \models n$ and subsets of $[n-1]$ given by sending α to

$$D(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{l-1}\}. \tag{9}$$

Given a sequence of integers $w = c_1 c_2 \ldots c_n$ we denote its *descent set* by

$$\operatorname{Des} w = \{i \ : \ c_i > c_{i+1}\}. \tag{10}$$

So every such w has an associated set $\operatorname{Des} w \subseteq [n-1]$ which corresponds to a composition α. In this case we define $F_w = F_\alpha$ and say that w *models* α. Note that one can tell by context whether the subscript is a word w or a composition α since the latter will have parentheses and commas while the former will not. Now suppose $\alpha \models m$ and $\beta \models n$. Let w_α and w_β be disjoint (as sets) and model α and β, respectively. In this situation, the multiplication of fundamental quasisymmetric functions is given by

$$F_\alpha F_\beta = \sum_{w \in w_\alpha \sqcup\!\sqcup\, w_\beta} F_w, \tag{11}$$

where the sum is over all ordinary shuffles of w_α and w_β.

Given α and a set C of positive integers with $|\alpha| = C$, there is a canonical way to construct a w modeling α with entries in C. Fill the cells of the diagram of α bijectively with the elements of C so that rows and columns increase to form a tableau T. So, in particular, if $C = [n]$, then T is a standard Young tableau of shape α. The *row word* of T, w_T, is constructed by concatenating the rows of T starting with the bottom row and moving up. Continuing the example started at the beginning of this section, we could take the tableau in Fig. 1 in which case $w_T = 568413927$. It is easy to see that the row and column restrictions on T imply that w_T models α.

We are now ready to put everything together and apply Takeuchi's formula. Fix once and for all a standard Young tableau T of shape α. Then

$$S(F_\alpha) = \sum_{k \geq 0} (-1)^k \sum_{\alpha_1 | \ldots | \alpha_k = \alpha} F_{w_{T_1}} \ldots F_{w_{T_k}}, \tag{12}$$

where the α_i are all nonempty and T_i is the subtableau cut out from T by α_i. To illustrate, suppose $\alpha = (2, 1)$ and

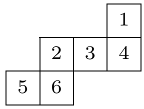
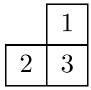
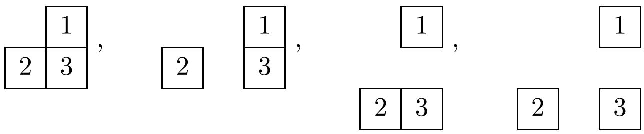
$T =$ 

Fig. 2. An example for F_α.

$T =$  .

Thus the terms in $S(F_{(2,1)})$ correspond to the decompositions



of T so that

$$S(F_{(2,1)}) = -F_{231} + F_2F_{31} + F_{23}F_1 - F_2F_3F_1.$$

We will adopt the terminology and notation of the previous section, the only differences being that we will use products of words to stand for products of fundamental quasisymmetric functions and that such products will be ordinary shuffles because of (11). We will use the notation α^t to stand for the *transpose* or *conjugate* of α, that is, the diagram obtained by reflecting α in the main diagonal.

Theorem 5.1 *([16]). The antipode in the fundamental basis of* QSym *is given by*

$$S(F_\alpha) = (-1)^{|\alpha|}F_{\alpha^t}.$$

Proof. Let $n = |\alpha|$. The definition of the splitting map is the same as in the proof of Theorem 4.1 except that one is dealing with products π of words and shuffles v which are terms in π. Also, in this case, we can apply the splitting map to any product with k factors where $k < n$ since n is the maximum number of nonempty subcompositions into which α can be decomposed by cuts. To illustrate, suppose $\alpha = (2, 3, 1)$ and we fix the tableau in Fig. 2. Now the product $\pi = 5 \cdot 6234 \cdot 1$ will contain the shuffle $v = 621345$. Since $j = 2$ is the smallest (in fact, only) index of a word in the product of length larger than one, we will have $\sigma(\pi, v) = (\pi', v)$ where $\pi' = 5 \cdot 6 \cdot 234 \cdot 1$.

The merge map is again very similar to the one used for the monomial quasisymmetric functions. We consider a shuffle v in the product $\pi = w_1 \cdot \ldots \cdot w_k$ and find the smallest index j, if any, such that

(i) $|w_1| = \cdots = |w_j| = 1$, and
(ii) if $B = w_jw_{j+1}$, then $v|_B$ is the row word of a rim-hook subtableau of T.

We then let $\mu(\pi, v) = (\pi', v)$ where

$$\pi' = w_1 \cdot \ \ldots \ \cdot w_{j-1} \cdot w_j w_{j+1} \cdot w_{j+2} \cdot \ \ldots \ \cdot w_k$$

which is well defined, as before, because of condition (ii). Taking as an example $\pi = 5 \cdot 6 \cdot 234 \cdot 1$ and $v = 621345$ we can not have $j = 1$ since then $B = \{5, 6\}$ and $v|_B = 65$ which is not the row word of the first two squares of T. But when $j = 2$ we have $x|_B = 6234$ which is the row word of the four middle squares of T. Thus $\mu(\pi, v) = (\pi', x)$ where $\pi' = 5 \cdot 6234 \cdot 1$, undoing the previous example's application of σ.

One now defines the involution exactly as was done for the M-basis, applying σ if possible and otherwise applying μ if possible. So the only thing new is to determine the fixed point(s) (π, v). As before, they will all be in the last summand of equation (12) which corresponds to $k = n$. So the factors in π are the individual elements of T and there is only one choice for π. Furthermore if μ can not be applied to (π, v), then one must have every pair of adjacent elements in w_T being in the reverse order in v. But this can only happen if v is w_T read backwards, which is precisely the row word of α^t. □

6. The fundamental basis of $\mathfrak{m}$QSym

Motivated by work of Buch [7] on set-valued tableaux, Lam and Pylyavskyy [15] defined six new Hopf algebras. These can be thought of as K-theoretic analogues of the symmetric function, quasisymmetric function, noncommutative symmetric function, and Malvenuto–Reutenauer Hopf algebras (the first and last both having two analogues). They appealed to Takeuchi's Theorem to conclude the existence of antipodes. Recently, Patrias [20] has given explicit formulas for these maps. We wish to show how one of these expressions can be derived using splitting and merging.

We will describe multi-QSym, denoted $\mathfrak{m}$QSym, the K-theoretic analogue of QSym. Because of the use of set-valued maps, we will need to permit arbitrary $\mathbb{Z}$-linear combinations of basis elements and these elements will not be homogeneous of a certain degree. Since elements of $\mathfrak{m}$QSym are not of bounded degree, it is not graded. However, we can still apply Takeuchi's formula because its proof also works more generally for any Hopf algebra where the projection map (1) is locally nilpotent.

Let $\tilde{\mathbb{P}}$ be the family of all finite, nonempty sets of positive integers. If $S, T \in \tilde{\mathbb{P}}$, then we write $S < T$ (respectively, $S \le T$) if $\max S < \min T$ (respectively, $\max S \le \min T$). Let $w = c_1 c_2 \ldots c_n$ be a permutation of $[n]$. A *w-set-valued partition* is a map $f : [n] \to \tilde{\mathbb{P}}$ satisfying

$$\begin{cases} f(i) \le f(i+1) & \text{if } c_i < c_{i+1}, \\ f(i) < f(i+1) & \text{if } c_i > c_{i+1} \end{cases}$$

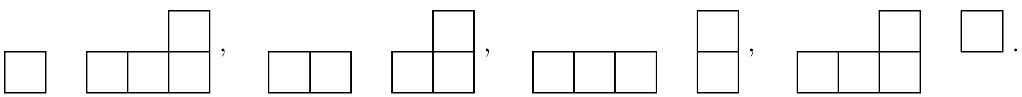
for $i \in [n-1]$. This is a special case of a more general definition for P-set-valued partitions, P a poset, which we will not need. To illustrate, if $w = 231$, then a w-set-valued partition would satisfy $f(1) \le f(2) < f(3)$. For example one could have $f(1) = \{5, 7\}$, $f(2) = \{7, 8, 10\}$ and $f(3) = \{11\}$.

Associate with any $S \in \tilde{\mathbb{P}}$ the monomial $\mathbf{x}_S = \prod_{s\in S} x_s$, and with any w-set-valued partition f the monomial $x_f = \prod_{i\in[n]} x_{f(i)}$. The *fundamental multi-quasisymmetric function* associated with a composition α is

$$\tilde{F}_\alpha = \tilde{F}_w = \sum_f x_f,$$

where w models α, and the sum is over all w-set-valued partitions f. Continuing the example above, $\alpha = (2, 1)$ is modeled by $w = 231$ and the given partition would contribute $x_5 x_7^2 x_8 x_{10} x_{11}$ to $\tilde{F}_{(2,1)}$. We note that the sum of the terms of least degree in $\tilde{F}_\alpha$ is exactly F_α. Finally we let $\mathfrak{m}$QSym be the span of the $\tilde{F}_\alpha$.

We need some combinatorial constructions to describe the bialgebra structure of $\mathfrak{m}$QSym. Many of the ideas which came into play in proving the antipode formula in the fundamental basis for QSym will also be used here. In addition to being able to separate a diagram at a cut-edge, we will need to be able to separate it at a cell. So if c is a cell of the diagram of α which we will call the *cut-cell*, then we write $\alpha = \beta \bullet \gamma$ where β is the composition whose diagram is all cells southwest of and including c and γ is the composition to the northeast and including c. Equivalently, α is formed by identifying the last square of β with the first square of γ. Note that both β and γ include c so that $|\beta| + |\gamma| = |\alpha| + 1$. For example, if $\alpha = (3, 1)$, then here are the various pairs β and γ as the cell c moves from southeast to northwest



The coproduct of $\mathfrak{m}$QSym can now be written

$$\Delta(\tilde{F}_\alpha) = \sum_{\beta,\gamma} \tilde{F}_\beta \otimes \tilde{F}_\gamma,$$

where the sum is over all β, γ such that $\alpha = \beta|\gamma$ or $\alpha = \beta \bullet \gamma$. Continuing our example

$$\begin{aligned}\Delta(\tilde{F}_{(3,1)}) = 1 \otimes \tilde{F}_{(3,1)} + \tilde{F}_{(1)} \otimes \tilde{F}_{(3,1)} + \tilde{F}_{(1)} \otimes \tilde{F}_{(2,1)} + \tilde{F}_{(2)} \otimes \tilde{F}_{(2,1)} + \tilde{F}_{(2)} \otimes \tilde{F}_{(1,1)} \\ + \tilde{F}_{(3)} \otimes \tilde{F}_{(1,1)} + \tilde{F}_{(3)} \otimes \tilde{F}_{(1)} + \tilde{F}_{(3,1)} \otimes \tilde{F}_{(1)} + \tilde{F}_{(3,1)} \otimes 1\end{aligned}$$

as the position of the cut travels over alternating edges and cells from southwest to northeast.

We can now apply Takeuchi's formula to get

$$S(\tilde{F}_\alpha) = \sum_{k\ge 0} (-1)^k \sum_{\alpha_1,\ldots,\alpha_k} \tilde{F}_{\alpha_1} \ldots \tilde{F}_{\alpha_k} \tag{13}$$

Fig. 3. The diagram of $(1) \bullet (2) \bullet (1)|(1)|(1) \bullet (1,1)$ and it superstandard labeling.

with the sum being over all $\alpha_1, \ldots, \alpha_k$ such that

$$\alpha = \alpha_1 \circ_1 \alpha_2 \circ_2 \cdots \circ_{k-1} \alpha_k, \tag{14}$$

where either $\circ_i = |$ or $\circ_i = \bullet$ for all i. Note that two expressions with the same α_i but different $\circ_i$ both contribute separately to the sum. For example, if $\alpha = (3)$, then $\alpha = (1) \bullet (2)|(1)$ and $\alpha = (1)|(2) \bullet (1)$ are different terms. We can write any expression of the form (14) by grouping together all the compositions between any two occurrences of an edge cut. Specifically, we will write

$$\alpha_1 \circ_1 \alpha_2 \circ_2 \cdots \circ_{k-1} \alpha_k = \beta_1|\beta_2|\ldots|\beta_m,$$

where $\beta_1 = \alpha_1 \bullet \cdots \bullet \alpha_a$, $\beta_2 = \alpha_{a+1} \bullet \cdots \bullet \alpha_b$, and so forth. We will call the β_i *components* and the α_j *subcomponents* of this expression. To illustrate, if $\alpha = (3,1,1)$, then

$$(3,1,1) = (1) \bullet (2) \bullet (1)|(1)|(1) \bullet (1,1)$$

has three components, namely $\beta_1 = (1) \bullet (2) \bullet (1)$, $\beta_2 = (1)$, $\beta_3 = (1) \bullet (1,1)$.

It will be convenient to have a geometric way to visualize the components and subcomponents of an expression. For the components, we will use the same convention as for QSym, where the diagram of α is split along the cut-edges. For the subcomponents, each cut-cell will be split in two by a vertical edge. These edges will be decorated with a bullet to distinguish them from the original edges of α. The diagram for the example at the end of the last paragraph is shown in Fig. 3. Note that the components of the expression are the connected components of the diagram while the subcomponents can be obtained by cutting each component along the bullet edges.

To complete our exposition, we need to describe how to take products of the $\tilde{F}_\alpha$. Given a word $w = c_1 \ldots c_n$, then a *multiword* on w is a word of the form $\tilde{w} = c_1^{m_1} \ldots c_n^{m_n}$ where $c_i^{m_i}$ indicates that c_i is to be repeated $m_i > 0$ times for all i. Exponents of one can be omitted. For example, if $w = 231$, then we could take $\tilde{w} = 2^4 3 1^2 = 2222311$. The *multishuffles* of words v and w, denoted $v \tilde{\sqcup\!\sqcup} w$, is the set of all words $x = d_1 \ldots d_r$ such that

1. x is a shuffle of some $\tilde{v}$ and $\tilde{w}$, and
2. $d_i \neq d_{i+1}$ for all $i \in [r-1]$.

Note that this set is infinite. By way of illustration if $v = 21$ and $w = 3$, then

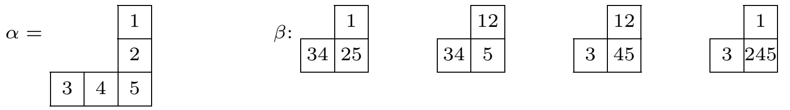


Fig. 4. Collapsing $\alpha = (3, 1, 1)$ onto $\beta = (2, 1)$.

$$21 \,\tilde{\sqcup\!\sqcup}\, 3 = \{213,\ 231,\ 321,\ 2321,\ 2131,\ 2313,\ 3213,\ 3231,\ \dots\},$$

where we have listed all the multishuffles of length 3 or 4. Now given compositions α and β we take words w_α and w_β modeling them, respectively, on disjoint alphabets. In this case

$$\tilde{F}_\alpha \tilde{F}_\beta = \sum_{w \in w_\alpha \,\tilde{\sqcup\!\sqcup}\, w_\beta} \tilde{F}_w. \tag{15}$$

Continuing our example, 21 models $\alpha = (1, 1)$ and 3 models $\beta = (1)$ so that

$$\tilde{F}_{(1,1)} \tilde{F}_{(1)} = \tilde{F}_{213} + \tilde{F}_{231} + \tilde{F}_{321} + \tilde{F}_{2321} + \tilde{F}_{2131} + \tilde{F}_{2313} + \tilde{F}_{3213} + \tilde{F}_{3231} + \dots\ .$$

To combine (13) and (15), consider a decomposition of the form (14). Label the corresponding diagram to form a tableau in a superstandard way with the numbers $1, \dots, \sum_i |\alpha_i|$ from left to right in each row starting with the top row and working down. Then, for each i, let w_i be the row word of the subtableau of T corresponding to α_i. Our decomposition will be represented by

$$\pi = w_1 \circ_1 \cdots \circ_{k-1} w_k \tag{16}$$

and the terms in the corresponding product of multi-quasisymmetric functions will be indexed by the multishuffles in $w_1 \,\tilde{\sqcup\!\sqcup} \dots \tilde{\sqcup\!\sqcup}\, w_k$. Fig. 3 illustrates these ideas, showing that $(1) \bullet (2) \bullet (1)|(1)|(1) \bullet (1, 1)$ when written in words becomes $4 \bullet 56 \bullet 7|8|2 \bullet 31$ which contains all terms of $S(F_{(3,1,1)})$ corresponding to the multishuffle $4 \,\tilde{\sqcup\!\sqcup}\, 56 \,\tilde{\sqcup\!\sqcup}\, 7 \,\tilde{\sqcup\!\sqcup}\, 8 \,\tilde{\sqcup\!\sqcup}\, 2 \,\tilde{\sqcup\!\sqcup}\, 31$.

To describe the coefficients of the cancellation free formula for the antipode we need the notion of collapsing a diagram. This operation is called merging in [20], but that would conflict with our use of the term in this work. We say that β is a *collapse* of α if one can successively collapse together boxes of α which share an edge to form β. We let $c_{\alpha,\beta}$ be the number of ways to collapse α to β. In counting collapses only the sets of boxes collapsed matters, not the order of the collapsing. Fig. 4 shows that if $\alpha = (3, 1, 1)$ and $\beta = (2, 1)$, then $c_{\alpha,\beta} = 4$. The labeling of the boxes is merely to show which sets were collapsed.

Theorem 6.1 *([20]). The antipode in the fundamental basis of* $\mathfrak{m}$QSym *is given by*

$$S(\tilde{F}_\alpha) = \sum_\beta (-1)^{|\beta|} c_{\beta,\alpha^t} \tilde{F}_\beta,$$

where the sum is over all compositions β.

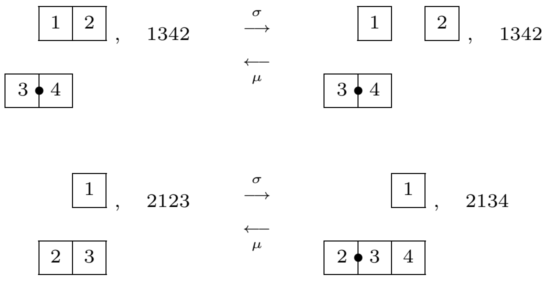


Fig. 5. The splitting and merging operations.

Proof. The proof parallels that of Theorem 5.1. All diagrams are labeled in a superstandard way and their reading words used in the corresponding multishuffles. We will define the involution on pairs (π, v) where π is a decomposition of the form (14) and v is a term in the multishuffle corresponding to π. We denote the image of the pair under the involution as (π', v'). As usual, the split operation is easiest to describe. For all i, let v_i be the subword of v which is a multiword on w_i where w_i is the reading word of α_i in the superstandard tableau for π. Find the smallest index j, if any, such that $|v_j| \geq 2$ and suppose $v_j = ab\ldots c$. There are now two cases.

1. If $a \neq b$, then let
   (a) $\pi' = \pi$ with α_j replaced by $(1)|\alpha'_j$ where α'_j is α_j with its first square removed, and
   (b) $v' = v$.
2. If $a = b$, then let
   (a) $\pi' = \pi$ with α_j replaced by $(1) \bullet \alpha_j$, and
   (b) $v' = v$ with one added to all elements greater than or equal to b except a.

To illustrate these cases, let $\alpha = (1, 2)$. Suppose $\pi = (1) \bullet (1)|(2)$ which in terms of words is $3 \bullet 4|12$ as can be seen in the top line of Fig. 5. Let $v = 1342$, a multishuffle in π with corresponding subwords $v_1 = 3$, $v_2 = 4$, and $v_3 = 12$. The only v_j with at least two elements is 12 and for this subcomponent $1 \neq 2$. So using the first case above, we have $\sigma(\pi, v) = (\pi', v')$ where $\pi' = (1) \bullet (1)|(1)|(1)$ and $v' = 1342$. On the other hand, suppose we consider $\alpha = (2, 1)$ and $\pi = (2)|(1)$ or in term of words $23|1$ as in the bottom line of Fig. 5. Then π contains the multishuffle $v = 2123$ with subwords $v_1 = 223$ and $v_2 = 1$. Now v_1 is the only subword with at least two elements and it begins with 22. So we are in the second case of the definition of σ. Thus $\pi' = (1) \bullet (2)|(1)$. Also v' is obtained from v by increasing the second 2 and all larger numbers by one to obtain $v' = 2134$. Note this convention is precisely what is needed to make v' a shuffle in π' and so this case is well defined. It is even easier to see that the first case is as well.

To describe the merge map, consider (π, v) and find the smallest index j, if any, such that $|v_i| = 1$ for $i \leq j$ and the concatenation $v_j v_{j+1}$ is a multisubword of the concatenation $w_j w_{j+1}$. Furthermore, if $\circ_j = \bullet$ we also insist that b and c are not consecutive in

v where $v_j = b$ and c is the first element of v_{j+1}. Again, there are two cases to define $\mu(\pi, v) = (\pi', v')$.

1. If $\circ_j = |$, then let
   (a) $\pi' = \pi$ with $\alpha_j|\alpha_{j+1}$ replaced by α'_j which is formed by edge-pasting α_j and α_{j+1} back together again, and
   (b) $v' = v$.
2. If $\circ_j = \bullet$, then let
   (a) $\pi' = \pi$ with $\alpha_j \bullet \alpha_{j+1}$ replaced by α'_jwhich is formed by cell-pasting α_j and α_{j+1} back together again, and
   (b) $v' = v$ with one subtracted from all element larger than or equal to c.

By way of example, consider $\pi = (1) \bullet (1)|(1)|(1)$ and $v = 1342$ which is the upper right pair in Fig. 5. In terms of words $\pi = 3 \bullet 4|1|2$ and v has corresponding subwords $v_1 = 3$, $v_2 = 4$, $v_3 = 1$, $v_4 = 2$. First consider $j = 1$. Then $v_1v_2 = 34$ is a subword of v. But $\circ_1 = \bullet$ and $3, 4$ are adjacent in v so they can not be merged. Next we try $j = 2$ giving $v_2v_3 = 41$. This is not a subword of v, so we test $j = 3$. Finally $v_3v_4 = 12$ is a subword of v and $\circ_3 = |$ so there is no further restriction. It follows that we can apply the first case of the merge definition and return to the original pair which started the split example. Now let us look at $\pi = (1) \bullet (2)|(1)$ and $v = 2134$ as in the lower right of Fig. 5. Translating to words gives $\pi = 2 \bullet 34|1$ and $v_1 = 2$, $v_2 = 34$, $v_3 = 1$. Taking $j = 1$ we see that $v_1v_2 = 234$ is a subword of v. Also $\circ_1 = \bullet$, but $2, 3$ are not adjacent in v so that we can merge as in case 2. This undoes the effect of σ in the second split example. It should not be hard for the reader to prove that σ is well defined.

We now define the involution ι exactly as in the case of QSym: applying σ if an appropriate index j can be found and otherwise applying μ. It is straightforward to verify that the minimality of j makes these two operations inverses. In fact case 1 for σ is the inverse of case 1 for μ and similarly for the 2nd cases. And, as usual, the fact that ι is sign-reversing is trivial.

There remains to determine the fixed points of ι. For this we will need a notion which is also useful in the theory of permutation patterns. Call a permutation v of the interval $[a, e]$ *colayered* if it is of the form

$$v = d\ (d + 1) \ldots e\ c\ (c + 1)\ \ldots (d - 1)\ \ldots a\ (a + 1) \ldots (b - 1)$$

for certain $a, b, c \ldots , e$. The reason for this terminology is because the complement of v is layered in the usual sense. The *layer lengths* of v are the cardinalities of the maximal increasing intervals. For example $v = 678945123$ is colayered with layer lengths $4, 2, 3$. There is a natural bijection between colayered permutations of $[d]$ and compositions of $\alpha = (\alpha_1, \ldots , \alpha_k)$ of d: just assign to each colayered permutation the composition of its layer lengths. For the inverse, take the diagram of α and let v be the row word of its

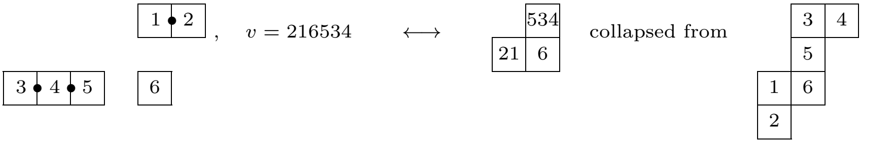


Fig. 6. A fixed point for $\alpha = (2, 1)$.

superstandard filling. So as v varies over all colayered permutations of $[d]$ the associated α will run over all compositions of d.

No let (π, v) be a fixed point of ι where $\pi = \alpha_1 \circ_1 \cdots \circ_{k-1} \alpha_k$. Such a fixed point for $\alpha = (2, 1)$ is shown in Fig. 6 to help clarify the following argument. Since we can not apply σ, we must have $|v_i| = 1$ for all i. This has several consequences. First of all v can have no repeated elements. Also, $|\alpha_i| = 1$ for all i. Thus every cut-edge of α (except the first and last) has been split to form π and there is a natural bijection between the cells of α and the components of π. Furthermore, every edge internal to a component is dotted. Let the reading words of the components of π be $w_1, \ldots, w_l$ in order from southwest to northeast. In the example, $w_1 = 345$, $w_2 = 6$, and $w_3 = 12$.

We claim that the possible second coordinates in our fixed point are exactly those of the form $v = w'_l w'_{l-1} \ldots w'_1$ where w'_i is a colayered permutation of the elements of w_i. And once this claim is established, we will be done. Indeed, recall the bijection between colayered permutations and compositions as well as the bijection between components of π and the cells of α. This combined with the order reversal in going from the w_i to the w'_i results in a bijection between the fixed points and row words of compositions β which collapse to α^t where the elements of v'_i collapse to form the ith box (counting from southeast to northwest) of α^t. See Fig. 6 for an illustration.

We will prove the claim in the case that the number of components is $l = 2$ since the general case is similar inducting on l. So let $w_1 = a_1 \ldots a_p$ and $w_2 = b_1 \ldots b_q$. Since the components are single cells, the subcomponents of a given component form a single row. It follows that the reading words w_1, w_2 are increasing sequences of consecutive integers. Since we can not apply μ we have, by case 1, that b_1 must be to the left of a_p in v. And by case 2, the only way that a_{i+1} can be to the right of a_i is if they are adjacent, with a similar statement holding for the b_j. It follows by induction on i and j that all the b_j must come before all the a_i. A similar induction shows that the a_i and b_j must be arranged in colayered permutations. This completes the proof. □

7. The incidence Hopf algebra on graphs

Now we turn our attention to the incidence Hopf algebra on graphs $\mathcal{G}$. We begin by introducing some notation. Let $G = (V, E)$ be a simple graph with vertex set $V = [n]$ and edge set E. We denote by $[G]$ the isomorphism class of G. The Hopf algebra $\mathcal{G}$ is the free $\mathbb{F}$-module on the isomorphism classes $[G]$. It has been studied by Schmitt [22], and Humpert and Martin [14] among others. Ardila and Aguiar (private communication) have recently described the antipode of $\mathcal{G}$ using the more general setting of Hopf monoids.

The product and coproduct maps in $\mathcal{G}$ are as follows

$$[G \cdot H] = [G \uplus H] \tag{17}$$

$$\Delta([G]) = \sum_{(V_1,V_2)\models[n]} [G_{V_1}] \otimes [G_{V_2}], \tag{18}$$

where G_{V_1} is the subgraph of G induced by the vertex set V_1 and similarly for G_{V_2}. We will henceforth drop the square brackets denoting equivalence class to simplify notation. No confusion should be caused by this convention.

To state the formula of Humpert and Martin for the antipode, we recall that a *flat* of G is a spanning subgraph F such that every component of F is induced. These are just the flats of the cycle matroid of G. We will denote the number of components of F by $c(F)$. Given any collection F of edges of G, the graph obtained from G by contracting the edges in F will be denoted G/F. Finally, we let $a(G/F)$ be the number of acyclic orientations O of G/F. That is, O is a digraph with underlying graph G/F such that O contains no directed cycles.

Theorem 7.1 *([14]). The antipode in $\mathcal{G}$ is given by*

$$S(G) = \sum_{F \text{ flat of } G} (-1)^{c(F)} a(G/F) F.$$

Proof. We will give a combinatorial proof of this theorem using the ideas from Section 2. By virtue of (17), (18), and Takeuchi's formula, the same reasoning that lead to equation (3) applied to $\mathcal{G}$ gives

$$S(G) = \sum_{k\geq 0} (-1)^k \sum_{(V_1,\ldots,V_k)\models[n]} G_{V_1} \uplus \cdots \uplus G_{V_k}. \tag{19}$$

Since each G_{V_i} is induced, each graph $F = G_{V_1} \uplus \cdots \uplus G_{V_k}$ in the inner sum is a flat of G. In order to show that the coefficient of F in $S(G)$ is $(-1)^{c(F)} a(G/F)$ we will construct, for each flat F, a sign-reversing involution ι on the set

$$A_F = \bigcup_{k\geq 0} \{\pi = (V_1,\ldots,V_k) \;:\; \pi \models [n] \text{ and } G_{V_1} \uplus \cdots \uplus G_{V_k} = F\}$$

with sign function

$$\operatorname{sgn} \pi = (-1)^k$$

for $\pi = (V_1,\ldots,V_k)$. This involution will have fixed points which are in bijection with acyclic orientations of G/F and which all have sign $(-1)^{c(F)}$, so we will be done.

We will first give the proof in the case that F is the flat with no edges and then indicate how this demonstration can be modified for a general flat. Note that if $\pi = (V_1,\ldots,V_k) \in$

A_F where F is the empty flat, then each V_i is a set of independent vertices. Let O_π be the orientation in $G = G/F$ defined by orienting its edges as

$$u \to v \text{ whenever } u \in V_i,\ v \in V_j \text{ and } i < j.$$

Note that O_π is acyclic because each V_i is independent and the usual ordering of the integers is transitive.

In general, there will be many different π giving rise to the same orientation. So given an orientation O, we let

$$\Pi_O = \{\pi \ :\ O_\pi = O\}.$$

Among all the partitions in Π_O, we distinguish a canonical one $\phi_O = (b_1^O, b_2^O, \ldots, b_n^O)$ defined as follows. Let b_1^O be the largest source in O. Next consider the digraph $O - b_1^O$ and let b_2^O be its largest source. Continue in this way, removing sources and selecting the largest source in the remaining graph until we are left with a single vertex b_n^O. An example follows this proof. Our strategy will be, for each acyclic orientation O, to construct a sign-reversing involution $\iota_O : \Pi_O \to \Pi_O$ which has ϕ_O as its unique fixed point. Note that $\operatorname{sgn} \phi_O = (-1)^n = (-1)^{c(F)}$ where F is the empty flat. This will complete the proof in the case under consideration.

To define $\iota = \iota_O$, suppose $\pi = (V_1, \ldots, V_k) \in \Pi_O$ with $\pi \neq \phi_O$. Then there must be a smallest vertex i where π and ϕ_O disagree, that is, $V_1 = b_1^O, \ldots, V_{i-1} = b_{i-1}^O$, but $V_i \neq b_i^O$. For simplicity, let $b = b_i^O$. Let V_j, $j \geq i$, be the block containing b. If $|V_j| \geq 2$, then we *split* V_j by replacing it with the ordered pair of blocks $V_j - b, b$ to obtain a partition $\sigma_j(\pi)$. If $|V_j| = 1$, then $i \neq j$ since otherwise π and ϕ_O would not differ at index i. Consider the suborientation O' obtained by restricting O to $V_i \uplus \cdots \uplus V_k$. So V_{j-1} is in O' which forces $V_{j-1} \uplus V_j$ to be independent: if there were an edge in the graph underlying O' from a vertex of V_{j-1} to $V_j = b$, then it would be oriented into b, contradicting the fact that b is a source in O'. So we can *merge* V_j by replacing it with $V_{j-1} \uplus V_j$ to form $\mu_j(\pi)$. Finally define

$$\iota(\pi) = \begin{cases} \sigma_j(\pi) & \text{if } |V_j| \geq 2, \\ \mu_j(\pi) & \text{if } |V_j| = 1. \end{cases}$$

As usual, it is clear that ι is sign-reversing. To check that it is an involution, we first show that the indices i for π and for $\iota(\pi)$ are the same. If a block is split, then b ends up one block to the right of its original position so that the ith block still differs from ϕ_O. If two blocks are merged, then the only way to change V_i is by merging in with V_{i+1}. But afterwards the ith block has size at least two and so again differs from that block in ϕ_O.

Since i does not change in passing from π to $\iota(\pi)$, the orientation O' must be the same for both. Thus b is also invariant under this map since it is the largest source in O'. Now the fact that ι is an involution follows in much that same way as in the proof for $\mathbb{F}[x]$. This finishes the demonstration for the empty flat.

We now deal with the general case. As already noted, every term in the sum (19) is a flat of G. Consider a flat F and a term $G_{V_1} \uplus \cdots \uplus G_{V_k} = F$. Then G/F is obtained by contracting every component of F to a point. Each such component lies in one of the G_{V_i}. So the partition $(V_1, \ldots, V_k)$ of $V(G)$ induces a partition $(W_1, \ldots, W_l)$ of $V(G/F)$ where each of the W_i is independent in G/F. Clearly this process is reversible with each partition of $V(G/F)$ into independent sets giving rise to a partition of $V(G)$ which induces the flat F. Now we can apply the same process as with the empty flat to the partitions of $V(G/F)$. This completes the proof. □

As an example of the involution in Theorem 7.1, consider the acyclic orientation O depicted in the first graph of Fig. 7. To compute the fixed point ϕ_O we first remove the largest source, which is vertex 5, to be the first component of ϕ_O. The largest source in what remains is 8 since the arc from 5 to 8 has been removed, and this becomes the second component of ϕ_O. Continuing in this way, we obtain

$$\phi_O = (5, 8, 7, 3, 6, 4, 2, 1)$$

as displayed in the second drawing of Fig. 7.

Now suppose we are given $\pi = (5, 3, 4, 26, 8, 7, 1)$ as in the third illustration of Fig. 7. It is easy to verify that $O_\pi = O$. Comparing π and ϕ_O, we see that they first differ in the second block so $i = 2$. The largest source of $O' = O - 5$ is $b = 8$ which is in the singleton block V_5. Thus

$$\iota(\pi) = \mu_5(\pi) = (5, 3, 4, 268, 7, 1)$$

as drawn at the end of Fig. 7.

Finally, consider $\pi' = \iota(\pi) = (5, 3, 4, 268, 7, 1)$. The reader should have no trouble verifying that for π' we still have $i = 2$ and $b = 8$ which is now in block 4. Since this block has other elements in it as well

$$\iota^2(\pi) = \iota(\pi') = \sigma_4(\pi') = (5, 3, 4, 26, 8, 7, 1) = \pi$$

as desired.

Viewed as maps on ordered partitions, the involutions for $\mathbb{F}[x]$ and for $\mathcal{G}$ are not the same. However, if we take G to be the graph with $V = [n]$ and no edges, then there is only one flat, namely $F = G$, and only one acyclic orientation O. So the involution in this case has a unique fixed point which is the same as the one in the proof of Theorem 2.1. In fact, we could have used this map to prove the polynomial result and emphasize even more the similarly of the demonstrations. We chose the earlier involution because of its simplicity.

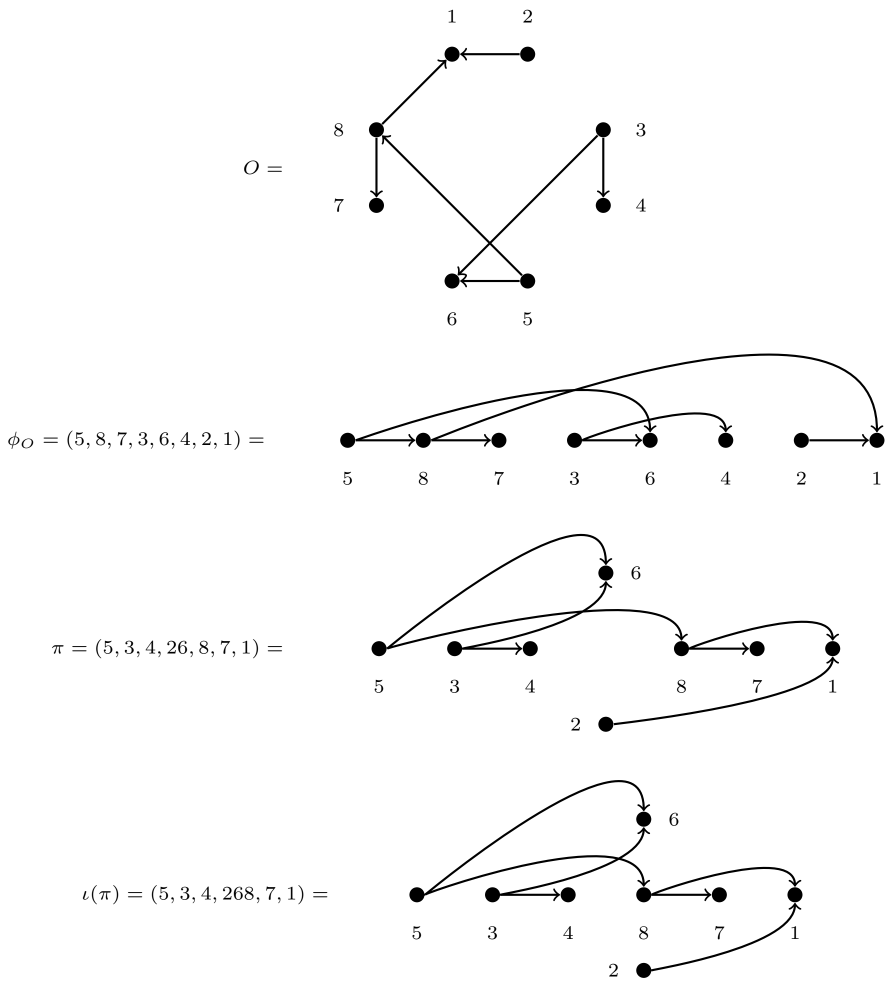


Fig. 7. Example for Theorem 7.1.

8. The immaculate basis for NSym

The Hopf algebra of noncommutative symmetric functions, NSym, was introduced by Gelfand et al. [10], and its immaculate basis, $\mathcal{S}_\alpha$, was defined in the paper of Berg et al. [5]. There is no known cancellation-free formula for the antipode acting on $\mathcal{S}_\alpha$ for general α. So in this section we derive expressions in the special cases where α is of hook shape or has at most two parts. In the case of hooks, the proof follows easily from the know expression for the antipode acting on the ribbon Schur basis. For shapes of at most two parts we give a new, cancellation-free expression proved by applying an involution.

The noncommutative symmetric functions are freely generated as an algebra by the noncommutative symbols H_n for n a positive integer. It is also convenient to let $H_0 = 1$ and $H_n = 0$ for $n < 0$. Similarly, the ordinary symmetric functions, Sym, are generated by the complete homogeneous symmetric functions $h_1, h_2, \dots$ which do commute. There is also the forgetful function NSym $\to$ Sym defined by algebraically extending the map $H_n \mapsto h_n$ for all n.

Bases for Sym are indexed by integer partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$, which are weakly decreasing sequences of positive integers. The all-important Schur function basis can be defined by the $k \times k$ Jacobi–Trudi determinant

$$s_\lambda = \det(h_{\lambda_i + j - i}).$$

Bases for NSym are indexed by compositions $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, which are arbitrary sequences of positive integers. To get a basis for NSym corresponding to the Schur functions, we define the noncommutative determinant of a $k \times k$ matrix $A = (a_{i,j})$ to be

$$\operatorname{Det} A = \sum_\sigma a_{1,\sigma(1)} a_{2,\sigma(2)} \dots a_{k,\sigma(k)},$$

where the sum is over all permutations σ of $[k]$. The *immaculate basis* for NSym is defined to be

$$\mathcal{S}_\alpha = \operatorname{Det}(H_{\alpha_i + j - i}).$$

If α is a partition, then we clearly have $\mathcal{S}_\alpha \mapsto s_\alpha$ under the forgetful map. To simplify the notation for products in this determinant we use, for any sequence of integers $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, the shorthand $H_\alpha = H_{\alpha_1} H_{\alpha_2} \dots H_{\alpha_k}$.

It is well known that Sym is actually a Hopf algebra where

$$\Delta h_n = \sum_{i=0}^{n} h_i \otimes h_{n-i}.$$

The antipode has a particularly nice action on the Schur basis, namely

$$S(s_\lambda) = (-1)^{|\lambda|} s_{\lambda^t}, \tag{20}$$

where λ^t is the conjugate of λ. We use the notation $|\lambda|$ and $l(\lambda)$ as we have done for compositions.

We also have that NSym is a Hopf algebra with

$$\Delta H_n = \sum_{i=0}^{n} H_i \otimes H_{n-i}. \tag{21}$$

But it appears as if the antipode is much harder to compute in the immaculate basis. So we will only derive formulas for it when α is a hook or when α has (at most) two rows.

We will first derive a formula for $S(\mathcal{S}_\alpha)$ when α is a hook. To do this, we recall another important basis of NSym. The *ribbon Schur function* corresponding to α is

$$R_\alpha = \sum_{\beta \geq \alpha} (-1)^{l(\alpha)-l(\beta)} H_\beta. \tag{22}$$

The antipode in the ribbon basis has a simple formula which can be found in the book of Grinberg and Reiner [13, Theorem 5.42].

Theorem 8.1. *For any composition α,*

$$S(R_\alpha) = (-1)^{|\alpha|} R_{\alpha^t}$$

A *hook* is a composition of the form $\alpha = (n, 1^k)$. In this case, the determinant for $\mathcal{S}_\alpha$ can be expanded around its first column since the second entry in that column is $H_0 = 1$ which commutes with the other H_i. This results in the recursion

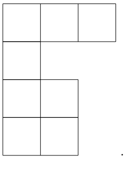
$$\mathcal{S}_{n,1^k} = H_n \mathcal{S}_{1^k} - \mathcal{S}_{n+1,1^{k-1}}.$$

On the other hand, partitioning the terms in the sum (22) for $R_{n,1^k}$ into those with $\beta_1 = n$ and those with $\beta_1 > n$ shows that this ribbon Schur function satisfies the same recursion. So the next result follows easily using Theorem 8.1 and induction on k. We note that Grinberg [12] has rederived this formula using his work on double posets, a concept introduced by Malvenuto and Reutenauer in [17].

Theorem 8.2. *For all $n \geq 1$ and $k \geq 0$,*

$$S(\mathcal{S}_{n,1^k}) = (-1)^{n+k} \mathcal{S}_{k+1,1^{n-1}}.$$

For the two-row case, we will express the antipode in terms of certain sets of tableaux. The *shape* of a composition $\alpha = (\alpha_1, \ldots, \alpha_l)$ is an array of l rows of left-justified boxes with α_i boxes in row i. We will use English notation where the first row is at the top as well as matrix coordinates for the cells. We also do not distinguish between a composition and its shape. So, for example,

(3,1,2,2) = 

A *dual immaculate tableau* of shape α is a placement T of positive integers in the cells of α such that the rows strictly increase and the first column weakly increases. The reason for using "dual" is because the strong and weak inequalities are interchanged from those for an immaculate tableau as defined in the paper of Berg et al. [5]. We write $\operatorname{sh} T = \alpha$. One dual immaculate tableau of shape $(3, 1, 2, 2)$ is

$$T = \begin{array}{lll} 1 & 3 & 4 \\ 1 & & \\ 2 & 6 & \\ 2 & 4 & \end{array}.$$

We let $T_c = T_{i,j}$ be the element of T in cell $c = (i, j)$. The *content* of T is the composition $\operatorname{co}(T) = (m_1, m_2, \dots)$ where m_i is the multiplicity of i in T. In our example tableau $\operatorname{co}(T) = (2, 2, 1, 2, 0, 1)$.

Suppose $\mathcal{T}$ is a set of tableaux. A *set of frozen cells* for $\mathcal{T}$ is a set of cells such that, for each such cell c, the element T_c is the same for all $T \in \mathcal{T}$. This includes the case when T_c is empty for all $T \in \mathcal{T}$. We will denote a frozen cell by giving its element a star. In the case the cell is to be empty, we use the symbol $\emptyset^*$. To illustrate, here is a set of tableaux indicating one of its sets of frozen cells

$$\mathcal{T} = \left\{ \begin{array}{lll} 1^* & 3^* & 4 \\ 1^* & 4 & \\ 2^* & 6^* & \emptyset^* \\ 2^* & & \\ \emptyset^* & & \end{array}, \quad \begin{array}{lll} 1^* & 3^* & 4 \\ 1^* & & \\ 2^* & 6^* & \emptyset^* \\ 2^* & 4 & \\ \emptyset^* & & \end{array}, \quad \begin{array}{lll} 1^* & 3^* & \\ 1^* & 4 & \\ 2^* & 6^* & \emptyset^* \\ 2^* & 4 & \\ \emptyset^* & & \end{array} \right\}.$$

In all cases of interest to us, the set of frozen cells will have the shape of a composition, that is, frozen cells in a row are left-justified and the set of frozen cells in the first column is connected. So we will call the elements in these cells a *frozen tableaux* T^*. Note that T^* includes the cells which are forced to be empty and we also require that all such cells are either at the right end of a row or at the bottom of the first column. In our example, the shape of T^* is $(2, 1, 3, 1, 1)$. A dual immaculate tableau which includes empty cells in this way will be called an *extended tableau.*

Now given an extended tableau T^* and a content vector v, define the set $\mathcal{T}(T^*, v)$ to be the set of all dual immaculate tableaux such that

(a) T^* is a frozen tableau for $\mathcal{T}(T^*, v)$,
(b) $\operatorname{co}(T) = v$ for every $T \in \mathcal{T}(T^*, v)$, and
(c) $\mathcal{T}(T^*, v)$ contains every dual immaculate tableau satisfying (a) and (b).

Note that our example $\mathcal{T}$ is such a set. It will turn out that the sets $\mathcal{T}(T^*, v)$ which we will need always also have the property

(d) if $v = (v_1, \dots, v_m)$, then $\operatorname{co} T^* = (v_1, \dots, v_{m-1}, w)$ for some $w \le v_m$.

So henceforth we also assume that $\mathcal{T}(T^*, v)$ also satisfies (d). It turns out that these are exactly the sets we need to describe the antipode.

Given $\alpha = (\alpha_1, \ldots, \alpha_l) \models n$ we will have to compute expressions of the form

$$S(H_{\alpha_1} H_{\alpha_2} \ldots H_{\alpha_l}) = S(H_{\alpha_l}) \ldots S(H_{\alpha_2}) S(H_{\alpha_1})$$
$$= (-1)^n \mathcal{S}_{1^{\alpha_l}} \ldots \mathcal{S}_{1^{\alpha_2}} \mathcal{S}_{1^{\alpha_1}},$$

where the second equality comes from the case $k = 0$ of Theorem 8.2. In [5, Theorem 7.3] a rule is given for expanding $\mathcal{S}_\alpha \mathcal{S}_\lambda$ in the immaculate basis whenever α is a composition and λ is a partition. Applying this rule repeatedly to the last equation easily gives

$$S(H_\alpha) = (-1)^{|\alpha|} \sum_T \mathcal{S}_{\mathrm{sh}(T)}, \tag{23}$$

where the sum is over all dual immaculate tableaux T with $\mathrm{co}(T) = (\alpha_l, \ldots, \alpha_2, \alpha_1)$. We are now in a position to prove our result for two-rowed tableaux. An example illustrating the following theorem follows its proof. One can also obtain a formula for $S(\mathcal{S}_{m,n})$ by changing to the ribbon basis, using Theorem 8.1, and changing basis back, but this expression in not cancellation free.

Theorem 8.3. *Given $m, n \geq 1$, let $\mathcal{T}_1 = \mathcal{T}(T_1^*, (n, m))$ and $\mathcal{T}_2 = \mathcal{T}(T_2^*, (n-1, m+1))$ where*

$$T_1^* = \begin{array}{ll} 1^* & \\ \vdots & \\ 1^* & 2^* \end{array}, \qquad T_2^* = \begin{array}{l} 1^* \\ \vdots \\ 1^* \\ \emptyset^* \end{array}.$$

Then

$$S(\mathcal{S}_{m,n}) = (-1)^{m+n} \sum_{T \in \mathcal{T}_1} \mathcal{S}_{\mathrm{sh}\, T} + (-1)^{m+n+1} \sum_{T \in \mathcal{T}_2} \mathcal{S}_{\mathrm{sh}\, T}.$$

Proof. Since $\mathcal{S}_{m,n} = H_{m,n} - H_{m+1,n-1}$ we see, using equation (23), that

$$S(\mathcal{S}_{m,n}) = (-1)^{m+n} \sum_T \mathcal{S}_{\mathrm{sh}\, T} + (-1)^{m+n+1} \sum_T \mathcal{S}_{\mathrm{sh}\, T},$$

where the first sum is over all possible dual immaculate tableaux of content (n, m) and the second over such tableaux of content $(n-1, m+1)$. Let $\mathcal{T}$ be the signed set which is the union of these two sets of tableaux, with signs being assigned so as to give the sums above. Now it suffices to find a sign-reversing involution $\iota : \mathcal{T} \to \mathcal{T}$ whose fixed points are exactly the tableaux in $\mathcal{T}_1 \cup \mathcal{T}_2$.

Consider $T \in \mathcal{T}$. If T contains n ones, then change the lowest one (which must be in the first column since T is dual immaculate) to a two as long as the resulting tableau is still dual immaculate. If T contains $n-1$ ones, then change the highest two in the first column of T (if one exists) to a one. Note that if this is possible, then the resulting tableau must be dual immaculate. If neither of these options is possible, then T is a fixed point.

It is clear from the definitions that this is an involution and reverses sign. To find the fixed points, note that the only tableaux with n ones fixed by ι are those where the lowest one also has a two in its row. This gives precisely the tableaux in $\mathcal{T}_1$. Similarly, the tableaux with $n-1$ ones fixed by ι are exactly those with no two in the first column which correspond to the tableaux in $\mathcal{T}_2$. This finishes the proof. $\square$

By way of illustration, to calculate $S(\mathcal{S}_{2,4})$ we compute

$$\mathcal{T}_1 = \left\{ \begin{matrix} 1^* & 2 \\ 1^* & \\ 1^* & \\ 1^* & 2^* \\ & \end{matrix}, \quad \begin{matrix} 1^* & \\ 1^* & 2 \\ 1^* & \\ 1^* & 2^* \\ & \end{matrix}, \quad \begin{matrix} 1^* & \\ 1^* & \\ 1^* & 2 \\ 1^* & 2^* \\ & \end{matrix}, \quad \begin{matrix} 1^* & \\ 1^* & \\ 1^* & \\ 1^* & 2^* \\ 2 & \end{matrix} \right\}$$

and

$$\mathcal{T}_2 = \left\{ \begin{matrix} 1^* & 2 \\ 1^* & 2 \\ 1^* & 2 \\ \emptyset^* & \end{matrix} \right\}$$

so

$$S(\mathcal{S}_{2,4}) = \mathcal{S}_{2,1,1,2} + \mathcal{S}_{1,2,1,2} + \mathcal{S}_{1,1,2,2} + \mathcal{S}_{1,1,1,2,1} - \mathcal{S}_{2,2,2}.$$

9. The Malvenuto–Reutenauer Hopf algebra

Aguiar and Sottile [2] were the first to give an explicit, combinatorial formula for the antipode of the Malvenuto–Reutenauer Hopf algebra of permutations, but it was not cancellation free. Aguiar and Mahajan [1] derived an expression for the antipode in this Hopf algebra using an antipode formula in a certain Hopf monoid. While their formula is cancellation-free, one needs the monoid structure for its construction. We will derive certain cancellation-free formulas which can be derived without appealing to the monoid. In particular, we will find such expressions for permutations whose image under the Robinson–Schensted map is a column superstandard Young tableau of hook shape. These tableaux will appear again in the next section when we consider the Poirier–Reutenauer Hopf algebra.

Let $\mathfrak{S}_n$ be the symmetric group on $[n]$ and $\mathfrak{S} = \cup_{n\geq 0}\mathfrak{S}_n$. The Malvenuto–Reutenauer Hopf algebra $\mathfrak{S}\mathrm{Sym}$ has basis $\mathfrak{S}$. To describe the product, if $\sigma = a_1 a_2 \dots a_n \in \mathfrak{S}_n$ and m is a positive integer, then let $\sigma + m$ denote the sequence obtained by increasing every element of σ by m. For example, if $\sigma = 231$, then $\sigma + 4 = 675$. Now if $\pi \in \mathfrak{S}_m$ and $\sigma \in \mathfrak{S}_n$, then we define

$$\pi \cdot \sigma = \sum_{\tau \in \pi \sqcup\!\sqcup (\sigma+m)} \tau$$

To illustrate

$$12 \cdot 21 = 1243 + 1423 + 1432 + 4123 + 4132 + 4312.$$

For the coproduct, we need the notion of standardization. If $\sigma = a_1 a_2 \ldots a_n$ is any sequence of distinct positive integers, then its *standardization* is the permutation $\mathrm{st}(\sigma)$ obtained by replacing the smallest a_i by one, the next smallest by two, and so on. By way of illustration $\mathrm{st}(9587) = 4132$. For $\pi \in \mathfrak{S}_n$ we let

$$\Delta(\pi) = \sum_{\sigma\tau=\pi} \mathrm{st}(\sigma) \otimes \mathrm{st}(\tau),$$

where $\sigma\tau$ represents concatenation of sequences, empty sequences allowed. As an example

$$\Delta(3142) = \epsilon \otimes 3142 + 1 \otimes 132 + 21 \otimes 21 + 213 \otimes 1 + 3142 \otimes \epsilon,$$

where ϵ is the empty permutation.

Each term in the Takeuchi expansion of $S(\sigma)$ is the sum of the elements of a set of shuffles $\sigma_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup \sigma_k$ and it will be convenient in what follows to identify the shuffle set with the sum of its elements.

The next result permits us to derive information about two antipode expansions at once. We write $[\pi]f$ for the coefficient of π in any formal sum of permutations f.

Theorem 9.1. *If $\pi, \sigma \in \mathfrak{S}_n$, then*

$$[\pi]S(\sigma) = [\sigma^{-1}]S(\pi^{-1}).$$

Proof. There is a bijection between the shuffle sets in $S(\sigma)$ and compositions α where the shuffle set $\sigma_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup \sigma_k$ corresponds to the composition $\alpha = (\alpha_1, \ldots, \alpha_k)$ with $|\sigma_i| = \alpha_i$ for all i. The sign of the shuffle set is $(-1)^k$ and π occurs at most once in each shuffle set. So to prove the theorem, it suffices to show that π occurs in the shuffle set of $S(\sigma)$ corresponding to α if and only if σ^{-1} appears in the shuffle set of $S(\pi^{-1})$ corresponding to α. By symmetry, it suffices to show the forward implication.

Suppose that π occurs in the shuffle set $\sigma_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup \sigma_k$. Then for all i we must have that σ_i is a subsequence of π. We will show that in this case the shuffle set $\pi'_1 \sqcup\!\sqcup \ldots \sqcup\!\sqcup \pi'_k$ in $S(\pi^{-1})$ corresponding to the same composition must contain a copy of σ^{-1} in that π'_i is a subsequence of σ^{-1} for all i. We will do the case $i = 1$ as the others are similar.

Suppose $\pi = a_1 \dots a_n$, $\sigma = b_1 \dots b_n$ and $\sigma_1 = \mathrm{st}(b_1 \dots b_l) = c_1 \dots c_l := \tau \in \mathfrak{S}_l$. Since σ_1 is a subsequence of π there must be indices $i_1 < \cdots < i_l$ such that $\pi(i_j) = c_j$ for $1 \le j \le l$. Because of this and the fact that $c_1, \dots, c_l$ is a permutation of $1, \dots, l$ it must be that $\pi'_1 = \tau^{-1}$. Also $\sigma(j) = b_j$ for $1 \le j \le l$ and $\sigma_1 = \tau$ implies that τ^{-1} is a subsequence of σ^{-1}. So π'_1 is a subsequence of σ^{-1} as desired. □

There is another way to use information about one value of the antipode map to determine a second. As in the theory of pattern avoidance, we consider the *diagram* of a permutation $\sigma \in \mathfrak{S}_n$ to be the set of points $(i, \sigma(i))$, $1 \le i \le n$, in the Cartesian plane. One can then ask, which of the eight operations on permutations induced by the action of the dihedral group of the square preserve coefficients of the antipode? Aside from the trivial action, there is only one other which leaves the multiset of coefficients invariant. Examples in $\mathfrak{S}_3$ and with $\sigma = 2413$ show that the other six actions do not preserve coefficients. Given σ, let σ^o be the permutation whose diagram is gotten from rotating the diagram of σ by 180 degrees. In other words if $\sigma = b_1 \dots b_n$, then

$$\sigma^o = (n + 1 - b_n) \dots (n + 1 - b_1).$$

The next proposition now follows easily from the fact that we always have $\mathrm{st}(w^o) = (\mathrm{st}\, w)^o$ and so we omit the proof.

Proposition 9.2. *If $\pi, \sigma \in \mathfrak{S}_n$, then*

$$[\pi]S(\sigma) = [\pi^o]S(\sigma^o).$$

Before we start to give formulae for the antipode on specific elements of $\mathfrak{S}\mathrm{Sym}$, we wish to recall an important connection with ribbon Schur functions. Consider the map $i : \mathrm{NSym} \to \mathfrak{S}\mathrm{Sym}$ defined by

$$i(R_\alpha) = \sum_{w \in \mathfrak{S}_n :\ \mathrm{Des}(w^{-1}) = D(\alpha)} w \tag{24}$$

where D and Des are as defined by (9) and (10), respectively. This is an injective Hopf algebra map [13, Corollary 8.14].

We now give some explicit, cancellation-free expressions for $S(\sigma)$ for various specific σ. We start with the identity permutation. This can easily be derived from Theorem 8.1 together with $i(R_n) = 12 \dots n$ and $i(R_{1^n}) = n(n-1) \dots 1$. But we prefer to give a merge-split proof.

Proposition 9.3. *We have*

$$S(12 \dots n) = (-1)^n (n(n-1) \dots 1).$$

Proof. It suffices to put a sign-reversing involution on the terms appearing in $S(12\dots n)$ whose unique fixed point is $\pi_0 = n(n-1)\dots 1$ with sign $(-1)^n$. The only shuffle set in $S(12\dots n)$ containing π_0 is the term $(-1)^n(1\sqcup\!\sqcup 2\sqcup\!\sqcup\dots\sqcup\!\sqcup n)$ which gives us the desired fixed point.

Now take any $\pi \neq \pi_0$. Then there must be a smallest index i such that $i+1$ appears to the right of i in π. Since the numbers $1,\dots,i-1$ appear in reverse order in π, every shuffle set in $S(12\dots n)$ containing π must be of the form

$$(-1)^k(1\sqcup\!\sqcup 2\sqcup\!\sqcup\dots\sqcup\!\sqcup(i-1)\sqcup\!\sqcup\sigma_i\sqcup\!\sqcup\dots\sqcup\!\sqcup\sigma_k)$$

for some k. If we are considering an appearance of π in a shuffle set with $|\sigma_i| = 1$, then let the involution pair it with the occurrence of π in the merged shuffle set

$$(-1)^{k-1}(1\sqcup\!\sqcup 2\sqcup\!\sqcup\dots\sqcup\!\sqcup(i-1)\sqcup\!\sqcup\sigma_i\sigma_{i+1}\sqcup\!\sqcup\sigma_{i+2}\sqcup\!\sqcup\dots\sqcup\!\sqcup\sigma_k).$$

If we are considering an appearance of π in a shuffle set with $|\sigma_i| > 1$, then let the involution pair it with an occurrence of π in the split shuffle set

$$(-1)^{k+1}(1\sqcup\!\sqcup 2\sqcup\!\sqcup\dots\sqcup\!\sqcup i\sqcup\!\sqcup\sigma_i'\sqcup\!\sqcup\sigma_{i+1}\dots\sqcup\!\sqcup\sigma_k),$$

where σ_i' is σ_i with i removed. It is clear from the definitions that these operations are inverses and sign-reversing. □

Using the Theorem 9.1, the previous proposition, and the fact that $12\dots n$ is its own inverse, we can compute when $12\dots n$ appears as a term in $S(\sigma)$ for any σ.

Corollary 9.4. *We have*

$$[12\dots n]S(\sigma) = \begin{cases} (-1)^n & \text{if } \sigma = n(n-1)\dots 1, \\ 0 & \text{else.} \end{cases}$$

Using similar techniques, we can prove the following result about $\sigma = n\dots 21$. It would be interesting to give general conditions under which $[\pi]S(\sigma) = [\pi']S(\sigma')$ where the prime denotes either reflection in a vertical axis or rotation by $\pi/2$ radians. Note that in either case $(12\dots n)' = (n\dots 21)$.

Proposition 9.5. *We have*

$$S(n\dots 21) = (-1)^n(12\dots n)$$

and

$$[n\dots 21]S(\sigma) = \begin{cases} (-1)^n & \text{if } \sigma = 12\dots n, \\ 0 & \text{else.} \end{cases}$$

We are going to generalize Propositions 9.3 and 9.5 to certain permutations starting with a decreasing sequence and ending with an increasing one. Applying the Robinson–Schensted map to these permutations outputs a pair of tableaux of hook shape the second of which is column superstandard. First let us introduce the notation

$$\eta_{k,l} = k(k+1)\dots l \quad \text{and} \quad \delta_{l,k} = l(l-1)\dots k$$

with the convention that if $k > l$, then $\eta_{k,l}$ and $\delta_{l,k}$ are both the empty word. We will further abbreviate to

$$\eta_k = \eta_{1,k} \quad \text{and} \quad \delta_k = \delta_{n,k}$$

when dealing with results for $\mathfrak{S}_n$. Another useful notion for applying induction is the following. If A is a set of positive integers with $|A| = n$ and $\mathfrak{S}_A$ is the set of permutations (linear orderings) of the elements of A, then we have the standardization bijection $\mathrm{st}_A : \mathfrak{S}_A \to \mathfrak{S}_n$. We then define, for any $\sigma \in \mathfrak{S}_A$,

$$S(\sigma) = \mathrm{st}_A^{-1} S(\mathrm{st}_A(\sigma)), \tag{25}$$

where st_A is extended linearly.

We will need the following lemma which is a refinement of the well-known fact that the alternating sum of any row of Pascal's triangle (except the first) is zero.

Lemma 9.6. *For any $n \geq 1$ we have*

$$\sum_{k=0}^{n} (-1)^k (\eta_k \sqcup\!\sqcup \delta_{k+1}) = 0.$$

Proof. It suffices to define a sign-reversing involution without fixed points on the terms appearing in the sum. Let v be a such a term and let k, $1 \leq k \leq n$, be the largest integer such that η_k is a subword of v. Then v appears in $(-1)^k(\eta_k \sqcup\!\sqcup \delta_{k+1})$ and, by maximality of k, in $(-1)^{k-1}(\eta_{k-1} \sqcup\!\sqcup \delta_k)$. Since these are the only two places v appears, v cancels and this is true for all v, leaving a sum of zero. □

Our next result is the promised generalization. Note that concatenation takes precedence over shuffle. Note also that the following formula is cancellation free since the terms in each summand end with a different integer.

Theorem 9.7. *For $1 \leq k < n$ we have*

$$S(\delta_{k,1}\eta_{k+1,n}) = \sum_{j=1}^{k} (-1)^{n+k+j} [\eta_{j-1} \sqcup\!\sqcup (\delta_{k,j+1} \sqcup\!\sqcup \delta_{k+2})(k+1)] j.$$

Proof. We will induct on k where Proposition 9.3 is the base case. We will show that the terms of $S(\delta_{k,1}\eta_{k+1,n})$ ending in j, $1 < j \leq k$, are as given in the summation. The cases $j = 1$ and $j > k$ are similar. Applying the definition of an antipode and (25) gives

$$S(\delta_{k,1}\eta_{k+1,n}) = -\left[\sum_{i=1}^{k} \delta_{i,1} \sqcup\!\sqcup S(\delta_{k,i+1}\eta_{k+1,n}) + \sum_{i=k+1}^{n} \delta_{k,1}\eta_{k+1,i} \sqcup\!\sqcup S(\eta_{i+1,n})\right].$$

Applying induction we see that the only terms ending in j will be in the first sum since $j \leq k$, and that these terms must come from the sum

$$\sum_{i=1}^{j-1}(-1)^{n+k+j+i-1}\left(\delta_{i,1} \sqcup\!\sqcup [\eta_{i+1,j-1} \sqcup\!\sqcup (\delta_{k,j+1} \sqcup\!\sqcup \delta_{k+2})(k+1)]j\right).$$

Extracting only the terms ending in j from this sum and factoring out the shared expression $v = (\delta_{k,j+1} \sqcup\!\sqcup \delta_{k+2})(k+1)$ gives

$$(-1)^{n+k+j}\left[\left\{\sum_{i=1}^{j-1}(-1)^{i-1}\delta_{i,1} \sqcup\!\sqcup \eta_{i+1,j-1}\right\} \sqcup\!\sqcup v\right] j.$$

We can now use Lemma 9.6 (reading all the words backwards for this application) to simplify the sum to η_{j-1}. Plugging in this as well as the value of v gives that the terms ending in j are exactly

$$(-1)^{n+k+j}[\eta_{j-1} \sqcup\!\sqcup (\delta_{k,j+1} \sqcup\!\sqcup \delta_{k+2})(k+1)]j$$

as desired. $\square$

Combining the previous result with Proposition 9.2, using the fact that

$$(\eta_j \sqcup\!\sqcup \delta_{j+1})^o = \delta_{n-j,1} \sqcup\!\sqcup \eta_{n-j+1,n}$$

and reindexing gives the following corollary.

Corollary 9.8. *We have*

$$S(\eta_{1,k}\delta_{n,k+1}) = \sum_{j=k+1}^{n} (-1)^{n+k+j+1} j[k(\delta_{k-1,1} \sqcup\!\sqcup \delta_{j-1,k+1}) \sqcup\!\sqcup \eta_{j+1,n}].$$

We end this section with a couple of conjectures. To state them, we will need to extend the previous notation. If A is a set of positive integers, then we let η_A and δ_A denote the increasing and decreasing words whose elements are A, respectively. Given $A \subseteq [n]$ we let $\overline{A} = [n] - A$ be the complement of A in $[n]$. If $a \in [n]$, then we use the abbreviation $\overline{a} = [n] - \{a\}$. We want to consider permutations of the form $\sigma_A = \delta_A\eta_{\overline{A}}$. Note that the previous theorem deals with the case when $A = [k]$ for some $k < n$. Now we consider what happens when $|A| = 1$. Note that when a summand in one of these expressions contains a number greater than n, then the expression is considered to be the empty sum. So, for example, the first summand in the next conjecture is empty if $n = 2$.

Conjecture 9.9. *Let $A = \{a\}$ where $1 < a \leq n$. We have*

$$S(\sigma_A) = (-1)^{n-1}(2 \sqcup\!\sqcup \delta_4)31 + (-1)^{n+a}((a-1)\eta_{a-2} \sqcup\!\sqcup \delta_{a+1})a + \sum_{j=2}^{a-1}(-1)^{n+j}[((j-1)\eta_{j-2} \sqcup\!\sqcup \delta_{j+1})j + ((j+1)\eta_{j-1} \sqcup\!\sqcup \delta_{j+2})j].$$

Here is what we believe happens when $|A| = 2$ and $2 \in A$.

Conjecture 9.10. *Let $A = \{a, 2\}$ with $2 < a \leq n$. We have*

$$S(\sigma_A) = (-1)^n[(32 \sqcup\!\sqcup \delta_5)41 + (12 \sqcup\!\sqcup \delta_4)3] + (-1)^{n-1}[(1 \sqcup\!\sqcup (3 \sqcup\!\sqcup \delta_5)4)2] + \sum_{j=3}^{a-1}(-1)^{n+j}[((j+1)21\eta_{3,j-1} \sqcup\!\sqcup \delta_{j+2})j - (j21\eta_{3,j-1} \sqcup\!\sqcup \delta_{j+2})(j+1)].$$

Note that all of the above shuffle expressions have coefficients ± 1 although when they are expanded as sums of permutations, the permutations can have larger coefficients. Can $S(\sigma_A)$ always be expressed in this form?

10. The Poirier–Reutenauer Hopf algebra

The Poirier–Reutenauer Hopf algebra [21], $\mathfrak{P}$Sym, is a sub-Hopf algebra of the Malvenuto–Reutenauer Hopf algebra. It has a distinguished basis indexed by standard Young tableaux. If $\pi \in \mathfrak{S}$, then let $P(\pi)$ denote the insertion tableau of π under the Robinson–Schensted correspondence. The basis element corresponding to a standard Young tableau P is defined to be

$$\mathbf{P} = \sum_{\pi \,:\, P(\pi)=P} \pi.$$

For example, if P is the tableau given in Fig. 8, then

$$\mathbf{P} = 32154 + 32514 + 32541 + 35214 + 35241.$$

We extend this notation in the obvious way to tableaux whose entries are not necessarily $1, \ldots, n$.

$$P = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad P' = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}$$

Fig. 8. Two standard Young tableaux, the first one superstandard, the second not.

Let $\mathfrak{P}\mathrm{Sym}$ be the span of the $\mathbf{P}$ as P runs over all standard Young tableaux. This is a graded Hopf algebra $\mathfrak{P}\mathrm{Sym} = \sum_{n\geq 0} \mathfrak{P}\mathrm{Sym}_n$ where the grading is inherited from $\mathfrak{S}\mathrm{Sym}$. The multiplication is given by

$$\mathbf{P} \cdot \mathbf{Q} = \sum_R \mathbf{R},$$

where the sum is over all standard Young tableaux R such that P is a subtableau of R and $Q = \mathrm{st}(j(R/P))$ where j is jeu de taquin and st is the standardization map applied to tableaux. If $|P| = n$ and Q is obtained by increasing all the entries of a standard Young tableau by n, then it will be convenient to also define $\mathbf{P} \cdot \mathbf{Q} = \mathbf{P} \cdot \mathrm{st}(\mathbf{Q})$. For example,

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \cdot \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \cdot \begin{array}{|c|c|}\hline 4 & 5 \\ \hline\end{array}$$

$$= \begin{array}{|c|c|c|c|}\hline 1 & 2 & 4 & 5 \\ \hline 3 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline 4 \\ \cline{1-1}\end{array}.$$

To describe the coproduct, let $\pi \cong \sigma$ mean that π and σ are Knuth equivalent. Then

$$\Delta(\mathbf{R}) = \sum_{P,Q} \mathrm{st}(\mathbf{P}) \otimes \mathrm{st}(\mathbf{Q}),$$

where the sum is over all P, Q whose row words satisfy $w_P w_Q \cong w_R$, or equivalently $P(w_P w_Q) = R$. As with the product, we will sometimes not standardize $\mathbf{P}$ and $\mathbf{Q}$. To illustrate, suppose

$$R = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

Then the words in the Knuth class of R are $\pi = 213$ and $\pi = 231$. So to compute $\Delta(\mathbf{R})$ we first look at all concatenations $\pi = \pi_1\pi_2$ where π_1 and π_2 are row words of tableaux. Putting a space between the prefixes and suffixes, we have $213 = \emptyset\ 213 = 2\ 13 = 21\ 3 = 213\ \emptyset$ where $\emptyset$ is the empty word and $231 = 2\ 31 = 23\ 1$. Note that we do not get $\emptyset\ 231$ since 231 is not the row word of any tableau. Translating from words to tableaux gives

$$\Delta(\mathbf{R}) = \emptyset \otimes \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} + \begin{array}{|c|}\hline 2\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|}\hline 3\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \times \emptyset$$
$$+ \begin{array}{|c|}\hline 2\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|c|}\hline 2&3\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline\end{array}$$
$$= \emptyset \otimes \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} + \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array} \times \emptyset$$
$$+ \begin{array}{|c|}\hline 1\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} \otimes \begin{array}{|c|}\hline 1\\\hline\end{array}.$$

The antipode in $\mathfrak{P}\mathrm{Sym}$ seems to be even more complicated than the one in $\mathfrak{S}\mathrm{Sym}$. But we at least have a result for certain hook-shaped tableaux. Let λ be a partition. The *column superstandard Young tableau of shape* λ, P_λ, is obtained by filling the first column with the numbers $1, \ldots, k$, then the second column with the numbers $k+1, \ldots, k+l$, and so forth. In Fig. 8, we have $P = P_{(2,2,1)}$ while P' is not column superstandard. Recall that the *descent set* of a standard Young tableau P, $\operatorname{Des} P$, is the set of all i such $i+1$ is in a lower row. From properties of the Robinson–Schensted correspondence it follows that $P(\pi) = P$ implies $\operatorname{Des} \pi^{-1} = \operatorname{Des} P$. If $\lambda = (n, 1^k)$ is a hook and $P = P_\lambda$ then the converse is also easily seen to be true. It follows that $\mathbf{P}_\lambda = i(R_{1^k,n})$ where i is the map in (24). Together with Theorem 8.1, this proves the following result.

Theorem 10.1. *If λ is a hook, then in $\mathfrak{P}\mathrm{Sym}$ we have*

$$S(\mathbf{P}_\lambda) = (-1)^{|\lambda|} \mathbf{P}_{\lambda^t}.$$

We have not been able to give an involution proof of this result except in some special cases. We have already done this when λ is a single row; see Proposition 9.3. It is also possible to use this technique on two-row hooks.

Proposition 10.2. *If $\lambda = (n-1, 1)$, then in $\mathfrak{P}\mathrm{Sym}$ we have*

$$S(\mathbf{P}_\lambda) = (-1)^{|\lambda|} \mathbf{P}_{\lambda^t}.$$

Proof. First note that the Knuth class for P_λ consists of the permutations

$$\pi_i = \eta_{2,i+1} 1 \eta_{i+2,n},$$

where $1 \leq i \leq n-1$. So the terms in Takeuchi's expansion for $S(\mathbf{P}_\lambda)$ are of the form $(-1)^k \mathbf{P}_1 \cdot \ldots \cdot \mathbf{P}_k$ where the $\mathbf{P}_j$ come from a concatenation of subwords of some π_i which are row words of tableaux. The variable k will always denote the number of factors. We will associate with such a term the pair (i, α) where π_i is the permutation giving rise to the term and $\alpha = (\alpha_1, \ldots, \alpha_k)$ is a composition where $\alpha_j = |P_j|$ for all j. Note that this pair fully determines the corresponding term. Finally, we will not standardize the P_j but rather write $\mathrm{st}(P_j)$ explicitly if we need to do so.

Initially, we will be canceling a single term in the expansion with a pair of terms which have the same sum but with opposite sign. The pair will be obtained by two different mergings of the single term. To describe the involution, it will be simplest to describe a bijection $f : D \to R$ whose domain elements are certain single terms and whose range elements are pairs of terms. Merging will correspond to applying f while splitting will correspond to applying f^{-1}.

Our first function $f_1 : D_1 \to R_1$ has a domain all terms with

$$\mathbf{P}_j = \boxed{1}$$

for some $j \neq k$. Note that it is also not possible to have $j = 1$ since none of the π_i begin with a one. Thus the associated pairs for these terms are of the form (i, α) where $i \neq n-1$ and $\alpha_j = 1$. The summands in the range will consist of all terms where 1 appears in a $\mathbf{P}'_j$ such that $j \neq 1$ and there is at least one other element in the tableau P'_j. Write $\mathbf{P}_1 \cdot \ \ldots \ \cdot \mathbf{P}_k = A \cdot \mathbf{P}_j \cdot \mathbf{P}_{j+1} \cdot B$ and let

$$f_1(A \cdot \mathbf{P}_j \cdot \mathbf{P}_{j+1} \cdot B) = A \cdot \mathbf{P}'_j \cdot B + A \cdot \mathbf{P}''_j \cdot B,$$

where the first summand is determined by the pair (i, α') and the second by $(i+1, \alpha')$ where α' is α with the parts $\alpha_j = 1$ and α_{j+1} replaced by a single part $\alpha'_j = 1 + \alpha_{j+1}$. For example, if $n = 5$, $i = 2$, and $\alpha = (1, 1, 1, 2)$, then $\pi_2 = 23145$ and the term corresponding to α is

$$\mathbf{P}_1 \cdot \mathbf{P}_2 \cdot \mathbf{P}_3 \cdot \mathbf{P}_4 = \boxed{2} \cdot \boxed{3} \cdot \boxed{1} \cdot \boxed{45} \ .$$

Now $j = 3$ and so $\alpha' = (1, 1, 1+2) = (1, 1, 3)$. So f maps this product to the sum of the products associated with α' in π_2 and $\pi_3 = 23415$ which is

$$f_1(\mathbf{P}_1 \cdot \mathbf{P}_2 \cdot \mathbf{P}_3 \cdot \mathbf{P}_4) = \boxed{2} \cdot \boxed{3} \cdot \boxed{1\,|\,4\,|\,5} + \boxed{2} \cdot \boxed{3} \cdot \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 4 & \\ \cline{1-1}\end{array} \ .$$

We first need to verify that f is well defined in that the subwords of π_i and π_{i+1} defined by α' are indeed row words of tableaux. Since 1 must be the subword of π_i corresponding to $\alpha_j = 1$, the word corresponding to α_{j+1} must be $\eta_{i+2,l}$ for some l. It follows that the subwords of π_i and π_{i+1} corresponding to $\alpha'_j = 1 + \alpha_{j+1}$ are $1\eta_{i+2,l}$ and $(i+2)1\eta_{i+3,l}$, respectively. It is easy to see that both of these are row words.

Next we need to show that a product and its image under f_1 cancel each other out in Takeuchi's expansion. Clearly the initial product and the two image products are of opposite sign. So it suffices to show that $\mathbf{P}_j \cdot \mathbf{P}_{j+1} = \mathbf{P}'_j + \mathbf{P}''_j$. Using the description of the corresponding subwords in the previous paragraph gives

$$\begin{aligned}
\mathbf{P}_j \cdot \mathbf{P}_{j+1} &= \begin{array}{|c|}\hline 1 \\ \hline\end{array} \cdot \begin{array}{|c|c|c|c|}\hline i+2 & i+3 & \dots & l \\ \hline\end{array} \\
&= \begin{array}{|c|c|c|c|c|}\hline 1 & i+2 & i+3 & \dots & l \\ \hline\end{array} + \begin{array}{|c|c|c|c|}\hline 1 & i+3 & \dots & l \\ \hline i+2 \\ \cline{1-1}\end{array} \\
&= \mathbf{P}'_j + \mathbf{P}''_j.
\end{aligned}$$

Finally, we need to prove that f_1 is a bijection. We do this by constructing its inverse. Suppose we are given a product such that the factor $\mathbf{P}'_j$ containing one satisfies $j \neq 1$ and $|P'_j| \geq 2$. We must find the product to add to the given one so that it can be mapped back to the domain. Suppose the product is associated with a pair (i, α'). Then considering the subwords of π_i which contain 1 and which are row words of tableaux, we see that there are only two possibilities for P'_j, namely

$$P'_j = \begin{array}{|c|c|c|c|c|}\hline 1 & i+2 & i+3 & \dots & l \\ \hline\end{array} \quad \text{or} \quad \begin{array}{|c|c|c|c|}\hline 1 & i+2 & \dots & l \\ \hline i+1 \\ \cline{1-1}\end{array}$$

for some l. If P'_j is the first (respectively, second) of these tableaux, then we pair the given product with the product associated with the pair $(i+1, \alpha')$ (respectively, $(i-1, \alpha')$). It is now easy to verify that adding this pair simplifies to a single product which was mapped to the pair by f_1.

The next part of the involution is similar to the first, so we will only provide a description of the map. Define a map $f_2 : D_2 \to R_2$ where the domain contains all products with

$$\mathbf{P}_1 = \begin{array}{|c|}\hline 2 \\ \hline\end{array} \quad \text{and} \quad \mathbf{P}_k = \begin{array}{|c|}\hline 1 \\ \hline\end{array}\ .$$

Note that this forces the term to be associated with the pair $(n-1, \alpha)$ where $\alpha_1 = \alpha_k = 1$. Let f_2 map this to the sum of the terms associated with the pairs $(1, \alpha')$ and $(n-1, \alpha')$ where α' is α with its first two parts replaced by $\alpha'_1 = 1 + \alpha_2$. One can then verify that the range contains sums of all pairs where the first (respectively second) summand contains one and two (respectively, two and three) in the first product tableau and n (respectively, 1) as a singleton in the last.

From the descriptions of D_1, D_2, R_1, R_2 and an examination of which subwords of the π_i can be row words, we see that the only terms which remain uncanceled so far are those associated with pairs of the form $(1, \alpha)$ where $\alpha_1, \alpha_k \geq 2$. Among these products, the only one which can produce $\mathbf{P}_{\lambda^t}$ is the one associated with $\alpha = (2, 1, 1, \dots, 1, 2)$. So we will be finished if we can define a sign-reversing involution on the tableaux $P \neq P_{\lambda^t}$ which occur in expanding the products under consideration.

Since all our products come from π_1, it suffices to specify α to focus on a given product whose expansion contains $\mathbf{P}$. So suppose the product corresponding to α has $\mathbf{P}$ as a term. Let l be the largest integer such that $1, \ldots, l$ are in the same cells in both P and P_{λ^t}. Equivalently we have $1, \ldots, l$ in the first column of P but $l+1$ is not. Since P comes from a product associated with π_1 and $\alpha_1 \geq 2$ we have $l \geq 2$. And since $P \neq P_{\lambda^t}$, $l \leq n-2$. Let $\mathbf{P}_j$ be the factor in the product containing l. If $l+1 \notin \mathbf{P}_j$, then $\mathbf{P}$ can be canceled with its appearance in the product corresponding to α' obtained from α by replacing α_j with $\alpha_j + \alpha_{j+1}$. If $l+1 \in \mathbf{P}_j$, then form α' by splitting $\alpha_j = \alpha'_j + \alpha'_{j+1}$ where α'_j corresponds to the prefix (respectively, suffix) of the α_j subword of π_1 consisting of those elements less than or equal to (respectively, greater than) l. Note that the bounds on l guarantee that α' will still satisfy $\alpha'_1, \alpha'_l \geq 2$. To illustrate, suppose $n=5$ and $\alpha=(2,3)$. Then $\pi_1 = 21345$ and so we are considering the product

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \cdot \begin{array}{|c|c|c|}\hline 3&4&5\\\hline\end{array} = \begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 2\\\cline{1-1}\end{array} + \begin{array}{|c|c|c|}\hline 1&4&5\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}.$$

Comparing the first summand to

$$\mathbf{P}_{\lambda^t} = \begin{array}{|c|c|}\hline 1&5\\\hline 2\\\cline{1-1} 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$$

we see that $l=2$. Since $l+1=3$ does not occur in the same tableau as 2 in the product, we merge and cancel this term with the tableau corresponding to $\alpha'=(5)$ which is

$$-\begin{array}{|c|c|c|c|}\hline 1&3&4&5\\\hline 2\\\cline{1-1}\end{array}.$$

For the second summand we have $l=3$ which is in the same tableau as $l+1=4$ in the product. So we split α into $\alpha'=(2,1,2)$ and see that this $\mathbf{P}$ will cancel with one of the terms in

$$-\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} \cdot \begin{array}{|c|}\hline 3\\\hline\end{array} \cdot \begin{array}{|c|c|}\hline 4&5\\\hline\end{array}.$$

The proof that this is a well-defined, sign-reversing involution is similar to previous arguments we have seen earlier and so is omitted. This completes the demonstration of the proposition. $\square$

11. Future work and open problems

We hope that this article will just be a first step in the exploration of the use of sign-reversing involutions to derive formulas for antipodes. In addition to the conjectures and questions already raised, here are three directions which would be interesting to explore.

1. Are there other Hopf algebras where the split or merge idea can be used to derive nice, preferably cancellation-free, formulas for S? Even more ambitious, is there a (meta)-involution which can be used to prove antipode identities for many different Hopf algebras at once? As was noted at the end of Section 7, one can use an involution for $\mathbb{F}[x]$ which is the special case of the one for $\mathcal{G}$ where the graph has no edges. We should also mention that the proof in Section 7 is based on discussions with Nantel Bergeron. The involution we introduce is closely related to the one of Bergeron and Ceballos [6] for their Hopf algebra of subword complexes. A formula for the antipode of a Hopf algebra of abstract simplicial complexes has been computed using these techniques by the first author in joint work with Hallam and Machacek [4]. Samantha Dahlberg (private communication) has used our method to compute the antipode for a Hopf algebra on involutions. Eric Bucher and Jacob Matherne [8] have used the merge-split technique to determine the antipode for the restriction-contraction Hopf algebra on uniform matroids. Finally, this method has come into play in the work of Bergeron and Benedetti [3] in their work on cancellation-free formulas for linearized Hopf monoids.
2. Can one obtain a full cancellation-free formula for the antipode in NSym in the immaculate basis? We attempted to at least do the three-row case, but the expressions in terms of frozen tableaux became increasingly complicated. However, there may be some other idea which is needed to unify all the cases.
3. We know from equation (20) that the antipode for Sym is particularly simple when expressed in terms of the Schur basis. Is there a way to derive this beautiful formula using a sign-reversing involution?

Acknowledgments

We are grateful to Nantel Bergeron and Darij Grinberg for enlightening discussions. Grinberg also found a proof of Theorem 10.1 while it was still a conjecture in a previous version of this paper. Finally, we thank the two anonymous referees for suggestions which improved the manuscript. In particular, one referee pointed out the connection of some of our results with ribbon Schur functions which was very useful.

References

[1] Marcelo Aguiar, Swapneel Mahajan, Monoidal Functors, Species and Hopf Algebras, CRM Monograph Series, vol. 29, American Mathematical Society, Providence, RI, 2010.

[2] Marcelo Aguiar, Frank Sottile, Structure of the Malvenuto–Reutenauer Hopf algebra of permutations, Adv. Math. 191 (2) (2005) 225–275.
[3] Carolina Benedetti, Nantel Bergeron, Cancelation free formula for the antipode of linearized Hopf monoid, preprint, arXiv:1611.01657.
[4] Carolina Benedetti, Joshua Hallam, John Machacek, Combinatorial Hopf algebras of simplicial complexes, SIAM J. Discrete Math. 30 (3) (2016) 1737–1757.
[5] Chris Berg, Nantel Bergeron, Franco Saliola, Luis Serrano, Mike Zabrocki, A lift of the Schur and Hall–Littlewood bases to non-commutative symmetric functions, Canad. J. Math. 66 (3) (2014) 525–565.
[6] Nantel Bergeron, Cesar Ceballos, A Hopf algebra of subword complexes, preprint, arXiv:1508.01465.
[7] Anders Skovsted Buch, A Littlewood–Richardson rule for the K-theory of Grassmannians, Acta Math. 189 (1) (2002) 37–78.
[8] Eric Bucher, Jacob P. Matherne, A cancellation-free antipode formula (for uniform matroids) for the restriction-contraction matroid Hopf algebra, preprint, arXiv:1612.04343.
[9] Richard Ehrenborg, On posets and Hopf algebras, Adv. Math. 119 (1) (1996) 1–25.
[10] Israel M. Gelfand, Daniel Krob, Alain Lascoux, Bernard Leclerc, Vladimir S. Retakh, Jean-Yves Thibon, Noncommutative symmetric functions, Adv. Math. 112 (2) (1995) 218–348.
[11] Ira M. Gessel, Multipartite P-partitions and inner products of skew Schur functions, in: Combinatorics and Algebra, Boulder, Colo., 1983, in: Contemp. Math., vol. 34, Amer. Math. Soc., Providence, RI, 1984, pp. 289–317.
[12] Darij Grinberg, Double posets and the antipode of QSym, preprint, arXiv:1509.08355.
[13] Darij Grinberg, Victor Reiner, Hopf algebras in combinatorics, preprint, arXiv:1409.8356.
[14] Brandon Humpert, Jeremy L. Martin, The incidence Hopf algebra of graphs, SIAM J. Discrete Math. 26 (2) (2012) 555–570.
[15] Thomas Lam, Pavlo Pylyavskyy, Combinatorial Hopf algebras and K-homology of Grassmannians, Int. Math. Res. Not. IMRN 2007 (2007) rnm125.
[16] Clauda Malvenuto, Christophe Reutenauer, Duality between quasi-symmetric functions and the Solomon descent algebra, J. Algebra 177 (3) (1995) 967–982.
[17] Claudia Malvenuto, Christophe Reutenauer, A self paired Hopf algebra on double posets and a Littlewood–Richardson rule, J. Combin. Theory Ser. A 118 (4) (2011) 1322–1333.
[18] Miguel A. Méndez, Jean Carlos Liendo, An antipode formula for the natural Hopf algebra of a set operad, Adv. in Appl. Math. 53 (2014) 112–140.
[19] Frédéric Menous, Frédéric Patras, Right-handed Hopf algebras and the preLie forest formula, preprint, arXiv:1511.07403.
[20] Rebecca Patrias, Antipode formulas for combinatorial Hopf algebras, preprint, arXiv:1501.00710.
[21] Stéphane Poirier, Christophe Reutenauer, Algèbres de Hopf de tableaux, Ann. Sci. Math. Québec 19 (1) (1995) 79–90.
[22] William R. Schmitt, Hopf algebras of combinatorial structures, Canad. J. Math. 45 (2) (1993) 412–428.
[23] Mitsuhiro Takeuchi, Free Hopf algebras generated by coalgebras, J. Math. Soc. Japan 23 (1971) 561–582.