$\mathbf{Ru} < \mathbf{E_7(5)}$ and $\mathbf{HS} < \mathbf{E_7(5)}$

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In this paper we describe our discovery that the sporadic simple groups Ru, HS and M_{22} are contained in the simple Chevalley group $E_7(5)$.

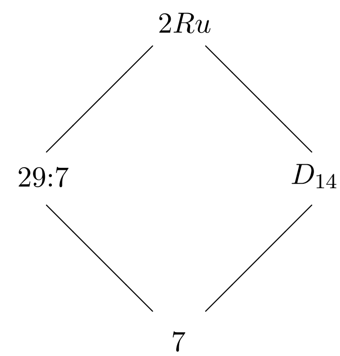
The work of [9] produces a short list of the possibilities for a sporadic simple subgroup of an exceptional group of Lie type. Apart from possible embeddings of M_{22}, HS and Ru in groups of type E_7 in characteristic 5, all of the embeddings of [9] are already known to occur. Thus our paper completes the classification of sporadic simple subgroups of exceptional groups of Lie type.

We offer two proofs of the embedding $Ru < E_7(5)$. The first is a computer proof, and the second is totally by hand. In particular, the second proof provides the only known computer-free construction of Ru. Moreover, our computer proof includes the first published presentation of Ru and thus gives the first easily verifyable computer construction of Ru. Similarly we give a hand proof and a computer proof of the embedding $HS < E_7(5)$. As a step in our hand proof of $HS < E_7(5)$ we establish the embedding $M_{22} < E_7(5)$: of course, since M_{22} is a subgroup of HS, this result also follows as a consequence of our computer proof of $HS < E_7(5)$.

We were led to conjecture the inclusions $Ru < E_7(5)$ and $HS < E_7(5)$ for the following reasons. The double cover $2.Ru$ has a faithful 28-dimensional character χ, and the character values of $\chi + \chi^*$ are all compatible with the character values of groups of type E_7 acting on their natural 56-dimensional module. Similarly the double cover $2.HS$ has a faithful 56-dimensional character, whose values are compatible with the character values of groups of type E_7 acting on their natural 56-dimensional module. Now $Ru, 2.Ru, HS$ and $2.HS$ all contain a subgroup $5^2{:}20$, an elementary abelian group of order 25 extended by a cyclic group of order 20 acting faithfully on the 5^2. Since 20 is not the order of an element in the Weyl group $W(E_7) = 2 \times S_6(2)$, it can be shown that $5^2{:}20$ does not embed in groups of type $E_7(\mathbf{K})$, where $\mathbf{K}$ is a field of characteristic prime to 5. Thus Ru and HS embed in $E_7(q)$ only if $5 \mid q$. On the other hand, all local subgroups of $2.Ru$ and $2.HS$ embed in $2.E_7(5)$, whence our conjectures $Ru < E_7(5)$ and $HS < E_7(5)$.

Throughout, G denotes the double cover $2.E_7(5)$, $\overline{G}$ denotes the simple group $E_7(5)$ and V is the natural 56-dimensional module for G over $GF(5)$. Most of our notation follows that of the ATLAS[At]. The four sections of our paper are independent and are arranged in chronological order. A later, independent, proof of $Ru < E_7(5)$ appears in [7].

Figure 1: Generating $2Ru$



1 A Computer Construction of Ru as a Subgroup of $E_7(5)$.

We used R. A. Parker's meataxe system (see [10]) to work with 56×56 matrices over $GF(5)$, representing G. Since Ru contains $2^6.G_2(2)$, where the $G_2(2)$ acts transitively on the 63 hyperplanes in the normal 2^6, it follows that any faithful representation of Ru over a field in odd characteristic has dimension at least 63. In particular Ru can not act on V and so $Ru \not\leq G$. Thus we seek to classify subgroups $2.Ru$ in G.

Any subgroup $2.Ru$ can be generated as in Figure 1. There is a unique class of groups of order 29 in G, with normalizer

$$(6 \times (29 \times 449).7).2 < (6 \times U_7(5)).2 < (2 \times U_8(5)).2.$$

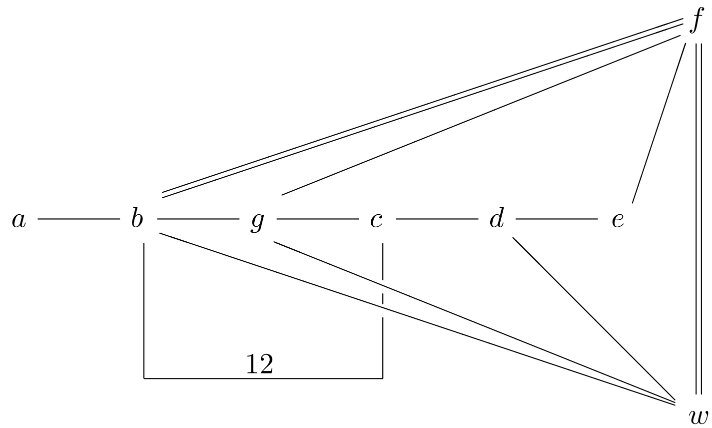
Moreover, an element of order 7 in $N_G(29)$ satisfies

$$C_G^*(7) = (SU_3(5).3 \times 7).2 \times SL_2(125).$$

(Here C^* denotes the invertilizer: that is the set of elements which either invert or centralize a given element.) An involution inverting the 7 extends $SU_3(5).3$ to $SU_3(5).Sym(3)$. There are precisely 9450 involutions in $SU_3(5).Sym(3) \setminus SU_3(5)$, and so the 7 is contained in precisely 2×9450 groups D_{14} in G. The factor of 2 comes from the involution in $SL_2(125)$, which lies in the center of G. Now G has just 3 classes of involutions, with representatives Z, X, ZX. Here $\langle Z \rangle = Z(G)$, and X and ZX have respective traces $+8$ and -8 on V. Thus we call the involutions conjugate to X and ZX plus involutions and minus involutions, respectively. Obviously the 2×9450 groups D_{14} come in pairs – one containing plus involutions, the other containing minus involutions.

Thus to classify subgroups isomorphic to $2.Ru$ in G, it suffices to check each of these 9450 plus D_{14}'s, and determine which ones together with the 29 generate $2.Ru$. On the computer, we found all 9450 such D_{14} groups, called $X_1, \ldots, X_{9450}$ say, and investigated the groups $Y_i = \langle 29, X_i \rangle$. We discarded any Y_i that contains an element with an order which is not the order of an element of $2.Ru$. Precisely 6 remained, $Y_1, \ldots, Y_6$ say. In order to identify these six groups, we use the presentation of Ru given by the following Theorem.

Figure 2: The Coxeter group X



Let X denote the Coxeter group with the Coxeter diagram given in Figure 2. (Thus X is generated by involutions a, b, c, d, e, f, g and w whose pairwise products have orders 2, 3, 4, or 12 according as the corresponding nodes of the diagram are unjoined, joined by an unmarked single edge, joined by a double edge, or joined by an edge marked 12.) Let R be the quotient of X obtained by adjoining the following additional relations: $c = g^{fed}$, $w = c^{gbag}c^{gabg}c$, $d = (bfg)^8$, $e = (bc)^6 = (abc)^4$, $(be)^d(be)^{dfbef} = db^a g^{bfabgbfg} = (ec)^{de}(ec)^{dw} = (cw)^{bcba}(cw)^{bwc} = c^{bcdewb}(db)^{wedcbcbcdewd} = 1$.

Theorem 1.1 *The group R is isomorphic to the Rudvalis group.*

Proof: Let T (respectively L) denote the group presented by those generators and relations of R that do not mention f or g, (respectively f, g, or w). Computer coset enumeration shows that the image of T in R has index 8120. Moreover, standard permutation group computations show that in the resulting permutation representation of R the images of T and L have sizes $11232 = |L_3(3){:}2|$ and $1600 \times |L_3(3){:}2| = |{}^2F_4(2)'|$. Moreover, the image of T has orbits of lengths 1, 1755 and 2304 in the permutation representation of R.

A second enumeration (of the cosets of the trivial subgroup in L) shows that $|L| = |L_3(3){:}2|$. It is routine to show that $L_3(3){:}2$ is a quotient of L, and thus the group L and its images in T and R are copies of $L_3(3){:}2$. A final coset enumeration shows that there are 1600 cosets of the image of L in T, and hence $|T| = |{}^2F_4(2)'|$. Our earlier computation of the size of the image of T in a permutation representation of R proves that the image of T in R has size $|{}^2F_4(2)'|$ and thus $|R| = 8120|T| = |Ru|$.

We now use a standard argument (see for example [17]) to show that the groups T and R must be simple. We illustrate this argument for the group T. The image of L in the permutation representation of T on the 1600 cosets of the image of L in T has orbits of sizes 1, 312, 351 and 936. In particular this faithful permutation representation of T must be primitive. Thus any minimal non-identity normal subgroup, N say, in T must be transitive on the cosets of the image of L. Moreover $N \cap L$ is isomorphic to a normal subgroup of $L_3(3){:}2$ and therefore $|N \cap L| \in \{1, |L_3(3)|, |L_3(3){:}2|\}$. We deduce that the characteristically simple group N has order 1600, $|{}^2F_4(2)'/2|$, or $|{}^2F_4(2)'|$. The classification of finite simple group shows that only the last of these orders is possible, and in particular T must be simple. Another appeal to the classification of finite simple groups gives $T \cong {}^2F_4(2)'$. A similar (slightly easier) argument now shows that R is simple and thus since the simple group Ru is characterized by its order, $R \cong Ru$. **QED**

We now apply Theorem 1.1 to identify the six subgroups $Y_1, \ldots, Y_6$ of G. For each of these groups we know generating matrices x, y, z of orders 29, 7, 2 (from the subgroups 29:7, 7 and D_{14} of Figure 1). We may assume that the element y is replaced by one of its powers so that $x^y = x^{16}$. Moreover, in each of the groups $Y_1, \ldots, Y_6$, experiment shows that we may replace x by one of its powers so that, modulo scalar matrices, xz has order 15 and xzx has order 20. (In fact there are two mutually inverse choices for x with these properties. We can make either choice of x.) We compute the following matrices in the group generated by x, y and z. Let $j = (x^3zxz)^{10}z$, $a = [z,j]^6$, $b = [z,j^2]^4$, $c = (b[z,j^3]^3)^6[z,j^3]^3$, $d = ([z,j^3]^3(ab^{j^2})^2)^2([z,j^3]^3(ab^{j^2})^2)^{z^j}$, $e = z$, $k = [a,x^4]^{(b[a,x^4]^5)^6ab}$, $l = c^{k^4}$, $f = l^{cd}l^{dc}l$, $g = c^{def}$, and $w = c^{gbag}c^{gabg}c$. Modulo scalar matrices, each of the groups $Y_1, \ldots, Y_6$ is generated by its elements a, b, c, d, e, f, g, w — since in each case the following words recover the generators x, y, z (modulo scalars): $z = e$, $m = aba(abgc)^5abg$, $y = [(mf)^{13}e(md)^{12}e(md)^{12}, e]$, $n = e[b,aw]^{(abgc)^5gc}$, $p = (ewg(abgc)^5a(abgc)^5bg(abgc)^5bawg(abgc)^5cbg)^{fw(ne^{fw})^3}$, $q = (pa)^7$, $r = (pb)^6$, $s = (pc)^{10}$, $x = (y^{-1}ps^rq^{rsqsqr})^{-2}$. Moreover, modulo scalar matrices, for each of the groups $Y_1, \ldots, Y_6$ the generators a, b, c, d, e, f, g, w satisfy the relations of R of Theorem 1.1. Therefore, $Z(G)Y_i/Z(G) \cong Ru$, for $i = 1, \ldots, 6$. Since we have already observed that Ru can not be a subgroup of G, we must have $Y_i \cong 2.Ru$ for $i = 1, \ldots, 6$.

Now $N_G(29) \cap N_G(7) \cong 12$ acts on $Y_1, \ldots, Y_6$. Furthermore, $N_{2.Ru}(29) \cap N_{2.Ru}(7) \cong 4$, and so each of $Y_1, \ldots, Y_6$ contains the subgroup of order 4 in $N_G(29) \cap N_G(7)$. Moreover, the group of order 3 in $N_G(29) \cap N_G(7)$ cannot normalize one of $Y_1, \ldots, Y_6$, for this group of order 3 centralizes the 29 (in G) and yet there is no group of order 3 in the 29−centralizer of $2.Ru$. Thus $N_G(29) \cap N_G(7)$ has two orbits of size 3 on $Y_1, \ldots, Y_6$. Now $2.Ru$ contains a unique class of 29:7, and we have

$$C^*_{2.Ru}(7) = Q_8 \times D_{14} \leq Q_8 \times Sz(8).$$

Thus the 7 is contained in precisely two groups D_{14}. Consequently each $2.Ru$ in G can be generated in just two ways as in (*), one way with a plus D_{14} and the other with the minus D_{14}. We have therefore proved that there are just two classes of $2.Ru$ in G.

Therefore, there are just two classes of subgroup isomorphic to Ru in $\overline{G}$. The non-abelian composition factors of centralizers of involutions in $\mathrm{Aut}(\overline{G})$ are $L_2(5)$, $O^+_{12}(5)$, $L_8(5)$, $U_8(5)$, $E_6(5)$, and ${}^2E_6(5)$, none of which contain Ru. Consequently the outer automorphism of $\overline{G}$ must interchange the two classes of Ru in $\overline{G}$.

We have now proved assertions (A) and (B), as well as part of (D), in the Theorem below. Our construction gives an explicit matrix action of $2.Ru$ on the natural 56-dimensional $GF(5)$-module for $2.E_7(5)$ and a standard application of the meataxe (see [10]) provides the decomposition of this module given in (D). The 133−dimensional adjoint module for the Lie algebra associated with $E_7(5)$ is a constituent of the symmetric square of the 56-dimensional module. We used the meataxe to determine that 133 is the smallest degree of a non-trivial constituent of the action of Ru on the symmetric square of the $56-$dimensional module. (The $56-$dimensional module has two irreducible $28-$dimensional constituents, thus we analyzed the two $406-$dimensional modules obtained as symmetric squares of $28-$dimensional representations, and the $784-$dimensional tensor product of the two $28-$dimensional representations. The last of these computations is close to the size limit for our implementation of the Meataxe.) The statements in (C) follow.

Theorem 1.2 *(A) The simple Chevalley group $E_7(5)$ contains precisely two classes of subgroups isomorphic to Ru.*

(B) The outer automorphism of $E_7(5)$ fuses the two classes.

(C) Each subgroup Ru acts irreducibly on the 133-dimensional Lie algebra associated with $E_7(5)$.

(D) In the double cover $2.E_7(5)$, each Ru lifts to $2.Ru$, and acts indecomposably with two irreducible constituents of dimension 28 on the natural 56-dimensional $GF(5)$-module for $2.E_7(5)$.

2 A Computer-free Construction of the Rudvalis Group as a Subgroup of $E_7(5)$.

In this chapter we will give a computer-free proof of the following Theorem.

Theorem 2.1 *Suppose that $E_6(5)$ has a subgroup ${}^2F_4(2)$ which acts irreducibly on the 27-dimensional $GF(5)E_6(5)$-modules, then $E_7(5)$ has a subgroup isomorphic to the Rudvalis group.*

We remark that an unpublished paper of M. Aschbacher on the maximal subgroups of E_6 contains a computer free proof that $E_6(5)$ indeed has a subgroup isomorphic to ${}^2F_4(2)$ which acts irreducibly on the $27-$spaces for $E_6(5)$.

Lemma 2.2 *Let X be a 2-dimensional vector space over $GF(5)$, S a Sylow 2-subgroup of $GL(X)$, A the unique subgroup of S isomorphic to $C_4 \times C_4$, ϕ an automorphism of S such that $X^\phi \cong X^*$, the dual module of X. Then ϕs inverts A for some s in S.*

Proof: Note first that there exist 1-dimensional subspaces X_1, X_2 of X with $X = X_1 \oplus X_2$, $S = N_{GL(X)}(\{X_1, X_2\})$ and $A = N_{GL(X)}(X_1) \cap N_{GL(X)}(X_2)$. Let ψ be the automorphism of A given by inversion. Then X^* and X^ψ are isomorphic as $A-$modules. Hence also X^ϕ and X^ψ are isomorphic as $A-$modules and there exists $s \in N_{GL(X)}(A))$ such that ϕs and ψ agree on A. Since $N_{GL(X)}(A) = S$, $s \in S$ and the lemma is proved. **QED**

Let E be the parabolic subgroup of G such that $E = QL$ with $Q = O_5(E)$, $|Q| = 5^{27}$ and $L \cong C_4 \times E_6(5)$. By assumption, L has ${}^2F_4(2)$ as a subgroup. But it can be proved that such a ${}^2F_4(2)$ cannot be extended to a $2.Ru$ in G. Instead we will look for a different class of complements with respect to Q. For this we first have to study the action of ${}^2F_4(2)$ on its 27-dimensional irreducible module over $GF(5)$.

Let F be a group with ${}^2F_4(2)' \leq F \leq C_4 \times {}^2F_4(2)$. Let S be a Sylow 2-subgroup of F and let P_1 and P_2 be the two maximal subgroups of F containing S, ordered so that $Z(P_1/Z(F)) \neq 1$. Let Γ_0 be the coset-graph of F with respect to P_1 and P_2. Then Γ_0 is the generalized octagon associated with ${}^2F_4(2)$. For $\gamma \in \Gamma_0$, let $\triangle^k(\gamma)$ be the set of vertices in Γ_0 at distance exactly k from γ. Further put $\alpha = P_1$ and $\beta = P_2$ and note that α and β are vertices of Γ_0. Put $T = F', L_i = P_i \cap T,\ Q_i = O_2(L_i),\ Z_i = Z(Q_i)$ and $V_1 = \langle {Z_2}^{P_1} \rangle$. Let $1 \neq z_1 \in Z_1$. If P_i normalizes a subgroup R_i in F or in some $F-$module, and if $\delta = P_i g \in \Gamma_0$, put $R_\delta = R_i^g$.

We assume that the reader is familiar with the structures of P_1, P_2 and S (see for example [11] or [5]). We remark here that $|S \cap T| = 2^{11}$, L_1/Q_1 is a Frobenius group of order 20, Z_1 has order 2, V_1/Z_1 is the unique irreducible L_1/Q_1 module of order 16, V_1 is elementary abelian, $L_2/Q_2 \cong Sym(3)$ and Z_2 is the unique irreducible L_2/Q_2-module of order 4.

We pay special attention to groups F such that

$$F/Z(F) \cong {}^2F_4(2),\ |Z(F)| = 2,\ F' \cong ({}^2F_4(2))' \quad \text{and} \quad F/F' \cong C_4. \tag{+}$$

Lemma 2.3 *Let W be a faithful irreducible 27−dimensional $GF(5)F$-module.*

(a) Let $U_1 = C_W(V_1)$.

(aa) $W = C_W(V_1) \oplus [C_W(Z_1), V_1] \oplus [W, Z_1]$,

(ab) $[C_W(Z_1), V_1] = \bigoplus_{\gamma \in \triangle^2(\alpha)} U_\gamma$,

(ac) $[W, Z_1] = \sum_{\gamma \in \triangle^4(\alpha)} U_\gamma = \bigoplus_{\gamma \in \triangle^3(\beta) \setminus \triangle^2(\alpha)} U_\gamma$,

(ad) $U_1 = C_W(V_1)$ is 1-dimensional, $C_W(V_1) = C_W(Q_1)$, $P_1/C_{P_1}(U_1) \cong C_4$ and U_1 is not isomorphic to its dual $GF(5)P_1$ module,

(ae) $[C_W(Z_1), V_1]$ is irreducible of dimension 10 and if ϕ is any automorphism of P_1, then $[C_W(Z_1), V_1]^\phi$ and the dual of $[C_W(Z_1), V_1]$ are not isomorphic as $GF(5)P_1-$modules,

(af) $[W, Z_1]$ is irreducible of dimension 16. If (+) holds $[W, Z_1]$ is isomorphic to its dual $GF(5)P_1-$module.

(b)

(ba) $W = C_W(Z_2) \bigoplus [W, Z_2]$.

(bb) $[W, Z_2] = \bigoplus_{\gamma \in \triangle^3(\beta)} U_\gamma$,

(bc) $C_W(Z_2) = \bigoplus_{\gamma \in \triangle^1(\beta)} U_\gamma$,

(bd) $[W, Z_2]$ is irreducible of dimension 24. If (+) holds $[W, Z_2]$ is isomorphic to its dual $GF(5)P_2-$module,

(be) $C_W(Z_2)$ is irreducible of dimension 3 and not isomorphic to its dual $GF(5)P_2$ module.

(c) Let $X \leq Y$ be $GF(5)P_1-$modules such that Y and Y/X are isomorphic to $[W, Z_1]$ as P_1-modules. Then Y splits over X.

(d) F' has two classes of involutions. If z_1 and i are representatives of these classes then $C_W(z_1)$ is 11-dimensional while $C_W(i)$ is 15-dimensional. Moreover, $|C_F(i)| = 2^{11} \cdot 3$.

(e) F has a unique class of elements d of order three such that $C_{F'}(d)$ has even order. Moreover, for any such d and any $t \in C_{F'}(d)$ with $|t| = 2$, we have $N_{F'}(D) \cap C_{F'}(t) \cong D_{24}$. If (+) holds then $|C_F(d)| = 2^5 \cdot 3^3$

(f) Let d be an element of order five in F. Then $C_F(d)$ is a $\{2, 5\}-$group.

Proof: (aa) and (ba) follow from the fact that V_1, Z_2 and Z_1 are $2-$groups and so coprime action applies.

Let $R_1 = [W, Z_1]$ and $\Sigma = \{H \leq V_1 \mid V_1 = Z_1 \oplus H\}$. Note that Q_1 and P_1 act transitively on Σ. By co-prime action

$$R_1 = \oplus_{H \in \Sigma} C_{R_1}(H).$$

It follows that $\dim R_1$ is a multiple of 16. Since $\dim W = 27$ we conclude that $\dim R_1 = 16$, hence $\dim C_{R_1}(H) = 1$ and thus Q_1 acts irreducibly on R_1. In particular the first half of (af) holds. Let $\gamma \in \triangle(\beta)$ with $\gamma \neq \alpha$. Since there exist exactly 8 elements of Σ containing Z_γ, we have $\dim C_{R_1}(Z_\gamma) = 8$. Further, $\dim C_W(Z_\gamma) = \dim C_W(Z_1) = 27 - 16 = 11$ and so

$$\dim C_W(Z_2) = \dim(C_W(Z_1) \cap C_W(Z_\gamma)) = 11 - 8 = 3.$$

Let $\Sigma_1 = \{H \leq V_1 \mid Z_1 \leq H, |V_1/H| = 2\}$ and $Y_1 = [C_W(Z_1), V_1]$. Then

$$Y_1 = \bigoplus_{H \in \Sigma_1} C_{Y_1}(H).$$

Note that P_1 has two orbits Σ_2 and Σ_3 on Σ_1 of lengths 5 and 10, respectively. So the dimension of Y_1 is a multiple of 5. Since $\dim [Y_1, Z_\gamma] = \dim [C_W(Z_1), Z_\gamma] = \dim [C_W(Z_\gamma), Z_1] = \dim C_{R_1}(Z_\gamma) = 8$ and $\dim C_W(Z_1) = 11$, we get $\dim Y_1 = 10$. Suppose that $C_{Y_1}(H) \neq 0$ for $H \in \Sigma_3$. Since Z_γ lies in exactly one member of Σ_2, the group Z_γ lies in exactly 6 members of Σ_3 and thus $C_{Y_1}(Z_\gamma)$ is 6-dimensional, a contradiction to $\dim C_{Y_1}(Z_\gamma) = 10 - 8 = 2$. So

$$Y_1 = \bigoplus_{H \in \Sigma_2} C_{Y_1}(H) \text{ and } \dim C_{Y_1}(H) = 2 \text{ for } H \in Z_2.$$

Recall that $U_1 = C_W(V_1)$. Then $C_W(Z_1) = U_1 \oplus Y_1$ and hence $\dim U_1 = 11 - 10 = 1$.

As W is irreducible, $U_1 \neq U_\gamma$, and so we have $[U_\gamma, V_1] \neq 1$. Observe that $C_{L_\gamma}(U_\gamma) \geq O^2(L_\gamma)$ (since L_γ acts as a subgroup of $GL_1(5) \cong C_4$ on U_γ). However, $V_1 \leq C_T(Z_\gamma) = L_\gamma$, and V_1 does not centralize U_γ. As $V_1 O^2(L_\gamma)/O^2(L_\gamma) \cong C_2$ is the unique proper subgroup of $L_\gamma/O^2(L_\gamma) \cong C_4$ we deduce that $C_T(U_\gamma) = C_{L_\gamma}(U_\gamma) = O^2(L_\gamma)$, thus $L_\gamma/C_{L_\gamma}(U_\gamma) \cong C_4 \cong P_\gamma/C_{P_\gamma}(U_\gamma)$ and hence U_1 is not self-dual as a $GF(5)P_1-$module.

Since V_1 inverts U_γ and Z_1 centralizes U_γ, U_γ lies in Y_1. Moreover, since $V_1 \leq L_\gamma$, the group V_1 acts on U_γ, thus $C_{V_1}(U_\gamma)$ is the hyperplane in Σ_2 that contains Z_γ. Now $C_{V_1}(U_\gamma) = V_1 \cap O^2(L_\gamma) = V_1 \cap Q_\gamma$. Moreover the single hyperplane of Σ_2 that contains Z_γ also contains

$Z_\gamma Z_1 = Z_\beta$ and so also contains $Z_{\gamma'}$, where $\triangle^1(\beta) = \{\alpha, \gamma, \gamma'\}$. Hence $V_1 \cap Q_\gamma = V_1 \cap Q_{\gamma'}$, and $\Sigma_2 = \{V_1 \cap Q_\delta \mid \delta \in \triangle^2(\alpha)\}$. So $C_{Y_1}(V_1 \cap Q_\gamma) = U_\gamma + U_{\gamma'}$, and

$$Y_1 = \bigoplus_{\delta \in \triangle^2(\alpha)} U_\delta.$$

In particular (ab) holds. Now $C_T(U_\gamma + U_{\gamma'}) = O^2(L_\gamma) \cap O^2(L_{\gamma'}) = Q_\gamma \cap Q_{\gamma'}$ and $|S \cap T/Q_\gamma \cap Q_{\gamma'}| = 32$. Since a Sylow $2-$subgroup of $GL_2(5)$ has order 32 and is isomorphic to C_4 wreath C_2 we conclude that $S \cap T/Q_\gamma \cap Q_{\gamma'} \cong C_4$ wreath C_2, that $S = (S \cap T)\, C_S\, (U_\gamma + U_{\gamma'})$ and that the action of S on $U_\gamma + U_{\gamma'}$ is irreducible, but not self-dual. Thus Y_1 is irreducible of dimension 10. Suppose there exists an automorphism ϕ of P_1 so that $[C_W(Z_1), V_1]^\phi$ is isomorphic to the dual module of $[C_W(Z_1), V_1]$. As $Frob_{20}$ has no outer automorphisms, we may assume without loss that ϕ centralizes $P_1/O_2(P_1)$. Then $(U_\gamma + U_{\gamma'})^\phi$ is isomorphic, as an $S-$module, to the dual module of $U_\gamma + U_{\gamma'}$. In particular, ϕ normalizes $C_S(U_\gamma + U_{\gamma'})$. Now $O_2(P_2)/C_S(U_\gamma + U_{\gamma'})$ is the unique subgroup of $S/C_S(U_\gamma + U_{\gamma'})$ isomorphic to $C_4 \times C_4$ and so by Lemma 2.2, up to an inner automorphism of S, the automorphism ϕ inverts $O_2(P_2)/C_S(U_\gamma + U_{\gamma'}))$. As $O_2(P_2)O_2(P_1) = S$ and $C_S(U_\gamma + U_{\gamma'}) \le O_2(P_1)$ we conclude that ϕ inverts $S/O_2(P_1) \cong C_4$, a contradiction to $[P_1, \phi] \le O_2(P_1)$. Thus no such ϕ exists and (ad) is proved.

Let $U_2 = C_W(Z_2)$. Then

$$U_2 = U_\alpha + U_\gamma + U_{\gamma'}.$$

and so (bc) holds. For $X \le P_2$, let $\tilde{X} = XC_{P_2}(U_2)/C_{P_2}(U_2)$. As $C_{L_2}(U_\alpha) = O^2(L_1) \cap L_2 = Q_1 = \bigcap_{\rho \in \triangle^1(\alpha)} L_\rho$, we have $C_{L_2}(U_2) = \bigcap_{\rho \in \triangle^2(\beta)} L_\rho$ and so by [5, 9.4.3]

$$\tilde{L_2} \cong (C_4 \times C_4).Sym(3)$$

In particular, (be) holds.

We will now determine $\tilde{P_2}$, which does depend on the precise structure of F. Since, by assumption, F is irreducible on W and since $C_W(Q_1)$ is $1-$dimensional, F acts absolutely irreducibly on W. Hence all elements of $Z(F)$ act as scalars on W and so also on U_2. Now no element of $\tilde{L_2}$ acts as a scalar and so $\tilde{Z(F)}\tilde{L_2} \cong Z(F) \times \tilde{L_2}$. As the full monomial subgroup of $GL(U_2)$ is isomorphic to $C_4 \times \tilde{L_2}$ we see that at least one of the following holds

- $Z(F) = 1$ and $\tilde{P_2} \cong (C_4 \times C_4).Sym(3)$
- $|Z(F)| \le 2$, $F \cong Z(F) \times {}^2F_4(2)$ and $\tilde{P_2} \cong C_2 \times (C_4 \times C_4).Sym(3)$
- $F/F' \cong C_4$ or $C_2 \times C_4$ and $\tilde{P_2} \cong C_4 \wr Sym(3)$

We remark that $\tilde{P_2}$ is uniquely determined by the structure of F except when $F \cong {}^2F_4(2)$. In this case its easy to see that F has two different irreducible $27-$dimensional representations giving rise to the two different possibilities for $\tilde{P_2}$.

Note that Q_1 is a normal subgroup of S generated by involutions. Furthermore we have $C_{S \cap T}(U_\gamma + U_{\gamma'}) \le Q_1$ and so $Q_1/C_{Q_1}(U_\gamma + U_{\gamma'})$ has order eight. Hence by the structure of $C_4 \wr C_2$, Q_1 acts as a D_8 on $U_\gamma + U_{\gamma'}$. In particular $O_2(P_1)$ acts irreducibly on $U_\gamma + U_{\gamma'}$.

Suppose in this paragraph that (+) holds. We wish to show that $U_\gamma + U_{\gamma'}$ is self–dual as an $O_2(P_1)-$module. Let B be the set of elements in P_2 that act as scalars on U_2, let $C = C_{P_2}(U_\gamma + U_{\gamma'})$, let $D = O_2(P_1) \cap O_2(P_2)$ and let $E = [O_2(P_2), S]$. As B is normal in P_2 with $O_2(P_2)/B \cong C_4 \times C_4$, [5, 9.4.3] implies $B \not\leq O_2(P_1)$ and so $B \not\leq D$. On the other hand, by the structure of $\tilde{S}$, there are exactly two subgroups $\tilde{X}$ of $\widetilde{O_2(P_2)}$ such that $\widetilde{Z(F)}\tilde{E} \leq \tilde{X}$ and $\widetilde{O_2(P_2)}/\tilde{X} \cong C_4$, namely $\tilde{C}\tilde{E}$ and $\tilde{B}\tilde{E}$. Since $S/O_2(P_1) \cong C_4$, the group $\tilde{D}$ meets the conditions on $\tilde{X}$ and so $\tilde{D} = \tilde{C}\tilde{E}$ and $D = CE$. But $E \leq Q_1 \cap Q_2$ and thus $D = C(Q_1 \cap Q_2)$. Since $O_2(P_1) = Q_1 D$ we conclude that $O_2(P_1) = CQ_1$. In particular, $O_2(P_1)$ acts as a D_8 and so self–dually on $U_\gamma + U_{\gamma'}$.

Back to the general case. Recall (ab) and put $Y_2 = \bigoplus_{\delta \in \triangle^2(\alpha) \setminus \triangle^1(\beta)} U_\delta$. Let $g \in L_1 \setminus L_2$. Then $P_2^g \cap S = O_2(P_1)$. Further $(U_\gamma + U_{\gamma'})^g$ is a Wedderburn component for $O_2(P_1)$ on Y_2 (and is self–dual if (+) holds). It follows that the $S-$module Y_2 is isomorphic to the $S-$module induced from the $O_2(P_1)-$module $(U_\gamma + U_{\gamma'})^g$ and is irreducible (and is self dual if (+) holds).

Pick $\delta \in \triangle^4(\alpha)$. By [5, 7.4,7.5], $Z_1 \leq L_\delta$ and $Z_1 \not\leq Q_\delta$. So Z_1 inverts U_δ and $U_\delta \leq [W, Z_1]$. As seen above Q_1 acts irreducibly on $R_1 = [W, Z_1]$. Further, Q_1 acts transitively on the subset $\triangle^3(\beta) \setminus \triangle^2(\alpha)$ of $\triangle^4(\alpha)$, $\dim[W, Z_1] = 16$ and $| \triangle^3(\beta) \setminus \triangle^2(\alpha)| = 16$. So

$$R_1 = [W, Z_1] = \bigoplus_{\delta \in \triangle^3(\beta) \setminus \triangle^2(\alpha)} U_\delta = \sum_{\delta \in \triangle^4(\alpha)} U_\delta.$$

Let $R_2 = [W, Z_2]$. From its definition, Y_2 is 8−dimensional and from our earlier calculation of $\dim[W, Z_2]$, the space $C_{[W,Z_2]}(Z_1)$ is 8−dimensional. Hence $Y_2 = [C_W(Z_1), Z_2] = C_{[W,Z_2]}(Z_1)$. So by coprime action, $R_2 = \bigoplus_{\delta \in \triangle^1(\beta)} C_{R_2}(Z_d) = Y_2 \oplus {Y_2}^d \oplus {Y_2}^{d^2}$, where d is an element of order 3 in L_2. It follows that

$$R_2 = \bigoplus_{\delta \in \triangle^3(\beta)} U_\delta.$$

Moreover, as Y_2 is irreducible (and self-dual if (+) holds) as an $S-$module, R_2 is irreducible (and self-dual if (+) holds) as a P_2-module. In particular, (bb) and (bd) hold.

So to complete the proof of (a) and (b), it remains to show that R_1 is self-dual as a P_1-module if (+) holds. Pick $H \in \Sigma$. Then $O^2(N_{P_1}(H))$ centralizes $C_{R_1}(H)$. As R_2 is self-dual as a P_2-module, S acts self-dually on R_2 and on R_1. This implies that $N_{P_1}(H) = SO^2(N_{P_1}(H))$ acts self-dually on $C_{R_1}(H)$. Since $R_1 = \bigoplus_{H \in \Sigma} C_{R_1}(H)$, the group P_1 is self-dual on R_1 and (a) and (b) are proved.

To prove (c) let $H \in \Sigma$. Then Y (respectively X) is induced from the 2 (1)-dimensional $N_{P_1}(H)$ module $C_Y(H)$, $(C_X(H))$. As Q_1 acts transitively on Σ, we have $N_{P_1}(H)Q_1 = P_1$. Thus $N_{P_1}(H)/O_2(N_{P_1}(H)) \cong Frob_{20}$ and $N_{P_1}(H)$ is generated by its 2-elements. Since $C_X(H)$ and $C_Y(H)/C_X(H)$ are isomorphic as $N_{P_1}(H)-$modules, all the 2−elements and so all the elements of $N_{P_1}(H)$ act as scalars on $C_Y(H)$. Hence $C_Y(H)$ splits over $C_X(H)$ and so also Y splits over X.

That F' has two classes of involutions is well known (see for example [3]). Clearly $C_W(z_1)$ is 11-dimensional. We can choose $i \in V_1$. Namely, choose $i \in V_1$ but $i \notin Z_\delta$ for $\delta \in \triangle^1(\alpha)$. Then i centralizes U_1, moreover i lies in exactly three elements of Σ_2 and in eight elements of Σ. Hence $\dim C_W(i) = 1 + 2 \cdot 3 + 8 = 15$ and so (d) holds.

(e) and (f) are well known and are easily deduced from [3]. **QED**

Lemma 2.4 *Let $K = D_8 \wr Sym(5)$ and let t be an automorphism of order 2 which centralizes $K/O_2(K)$ but not $O_2(K)/Z(O_2(K))$. Let A be a non-abelian subgroup of $O_2(O^2(K))$ such that $|N_K(A)|$ is divisible by five and $Z(K) \not\leq A$. Then $|A| = 2^8$, $N_{K\langle t\rangle}(A)/A \sim D_8.Frob_{20}$ and $N_{K\langle t\rangle}(A)$ induces an outer automorphism on $O_2(N_{K\langle t\rangle}(A))/A$.*

Proof: Let x be an element of order five in K normalizing A. Put $X = \langle x\rangle$, $I = O_2(O^2(K))$ and $J = [Z(I), K]$. Then $I/Z(K)$ has order 2^{12}, the group X acts fixed point freely on $I/Z(K)$, also $|J| = 2^4$ and $Z(I) = I' = Z(K)J$. As A is not abelian and $Z(K) \not\leq A$, $A' = J = Z(A)$. Also $AZ(I) \neq I$ since otherwise $Z(K) \leq I' \leq A' \leq A$. Hence $|A/J| = 2^4$ and $|A| = 2^8$. Let $A^* = AZ(I)$ and note that $\Phi(A^*) = \Phi(A) = J$. Let $D_1, \ldots, D_5$ be the five $D_8's$ that are naturally permuted by K, ordered so that $D_{i+1} = D_i^x$. Let $D_1 = \langle a_1^*, b_1^*\rangle$, with a_1^* and b_1^* of order two. Inductively define $a_{i+1}^* = a_i^x$ and $b_{i+1}^* = b_i^x$. Let $z_i^* = [a_i^*, b_i^*] \in Z(D_i)$. For $c \in \{a, b, z\}$ let c_i be the product of the c_j^* with $1 \leq j \leq 5$ and $j \neq i$. If $a_1 \in A^*$ then $A^* \leq \langle a_1^X\rangle Z(I) = \langle a_1, \ldots a_5\rangle Z(I)$ and A^* is abelian, a contradiction. Similarly $b_1 \notin A^*$ and $a_1b_1 \notin A^*$. Thus A^* has an element s of the form a_1b_i with $2 \leq i \leq 5$ or of the form $a_1b_ib_j$ with $1 \leq i < j \leq 5$. Note that $(a_1b_2)^2 = [a_1, b_2] = z_3^*z_4^*z_5^* \notin J$ and so the case $s = a_1b_2$, and more generally the case $s = a_1b_i$, is impossible. Note also that $(a_1b_1)^2 = z_1 \in J$. So we get $i \neq 1$ in the second case. Suppose that $i = 2$ and $j = 3$. Then $s^x = a_2b_3b_4$ and $[s, s^x] = [a_1, b_2b_3][b_2, a_2][b_3, a_2]$. Since the first two of these factors are in J but the last one is not, this case is impossible. Similarly, the cases $(i, j) = (2, 4), (3, 5)$ and $(4, 5)$ are ruled out. Thus $(i, j) = (2, 5)$ or $(3, 4)$. Since $[A^*, X] = A$, there exist exactly two choices for A for a given X. Note that an element in the normalizer of X which acts as (2354) on $D_1, \ldots, D_5$ interchanges these two choices. Let $L = K\langle t\rangle$. Then $|N_L(X)/N_L(X) \cap N_L(A)| = 2$. Let $D_0 = C_{O_2(K)}(X)$. Then $[D_0, I] \leq J$ and so $D_0 \leq N_K(A)$ and $N_{O_2(K)}(A) = D_0A$. Since $O^2(K)$ does not normalize A^* we conclude that $N_L(A) \leq N_L(X)O_2(K)$ and we have to decide whether $N_L(A)/D_0A$ is isomorphic to $C_2 \times D_{10}$ or to $Frob_{20}$. In the first case we may assume that t normalizes A. As X acts irreducibly on A/J we conclude that $[A, t] \leq J$ and so $A^*/Z(I) = C_{I/Z(I)}(t)$. But then K normalizes $A^*/Z(I)$, a contradiction. The very last statement follows as $N_{K\langle t\rangle}(A) \not\leq K$. **QED**

Lemma 2.5 *(a) $E_6(5)$ has two conjugacy classes of involutions. There exist representatives r and s of these classes such that*

$$C_{E_6(5)}(r) \sim 4.D_5(5).4 \quad \textit{and} \quad C_{E_6(5)}(s) \sim 2.(L_2(5) \times L_6(5)).2.$$

Moreover, if U is a 27-dimensional $GF(5)-$module for $E_6(5)$, then $C_U(r)$ is 11-dimensional and $C_U(s)$ is 15-dimensional. As a $C_{E_6(5)}(r)-$module U is the direct sum of three irreducible submodules of dimensions 1, 10 and 16. The kernel of the action of $C_{E_6(5)}(r)$ on the invariant 1-space is $C_{E_6(5)}(r)'$.

(b) $2.E_7(5)$ has three conjugacy classes of involutions. There exist representatives z, z_0 and i of these classes such that $z_0 \in Z(2.E_7(5))$, $i = z \cdot z_0$ and

$$C_{2.E_7(5)}(z) \sim 2^2.(L_2(5) \times D_6(5)).2$$

Moreover, $C_V(z)$ is a tensor product of natural modules for $SL_2(5)$ and $\Omega_{12}^+(5)$, and $[V, z]$ is a half-spin module for $2.D_6(5)$.

Proof: The conjugacy classes of involutions and their centralizers are well known and easily deduced, for example by the methods found in [2]. The information about U and V is easily computed using the subgroup $5^{27}E_6(5)$ of $E_7(5)$, the subgroup $5^{1+56}2E_7(5)$ of $E_8(5)$, the Steinberg relations and the weight theory of modules for groups of Lie type. **QED**

Recall that E is a parabolic subgroup $QL \sim 5^{27}.(4 \times E_6(5))$ in G. Let $X_1 = C_V(Q)$, $X_3 = [V, Q]$ and $X_2 = [X_3, Q]$. Then

$$0 < X_1 < X_2 < X_3 < V$$

is the unique chief series for E on V, moreover $E = N_G(X_1)$, the modules X_1 and V/X_3 are 1-dimensional and X_2 and X_3/X_2 are 27−dimensional mutually dual E−modules.

By assumption L' contains a subgroup ${}^2F_4(2)$ acting irreducibly on Q. Hence L contains a subgroup $\tilde{F}$ fulfilling (+), and acting irreducibly on Q.

Define $\widetilde{S}, \widetilde{P}_1$ and $\widetilde{P}_2$ (as subgroups of $\tilde{F}$) in a way analogous to the above. The reader should notice that $Z(\tilde{F}) = Z(G)$ centralizes Q, thus Q is irreducible, but not faithful as an $\tilde{F}$−module. But we still can apply Lemma 2.3 to $\tilde{F}/Z(G)$ and conclude that $\widetilde{P}_1$ normalizes a non-trivial subgroup $\langle f_- \rangle$ of Q. Let $f_+ \in \widetilde{P}_1$ be of order 5 and $r \in \widetilde{P}_1 \cap \widetilde{F}'$ of order 4 with ${f_+}^r = {f_+}^2$. As seen in the proof of Lemma 2.3, $\langle r \rangle$ acts faithfully on $\langle f_- \rangle$. So either ${f_-}^r = {f_-}^2$ or ${f_-}^r = {f_-}^3$. Note that there exists σ in $N_G(L)$ inducing a graph automorphism on L (indeed such a σ can be chosen to invert the Cartan subgroup of G). Then the action of L on Q^σ is dual to the action of L on Q and replacing $\tilde{F}$ and f by $\tilde{F}^\sigma$ and f^σ if necessary, we may assume that ${f_-}^r = {f_-}^2$. Put $f = f_+ f_-$. Then $f^r = f^2$. Let u be an element of order 4 in $\widetilde{P}_1$ centralized by f_+. Let $Z_1 = Z(\widetilde{P}_1 \cap \widetilde{F}')$.

Lemma 2.6 *(a) The element u centralizes f_-.*

(b) f_- is a root element of G and $C_G(f_-) \sim 5^{1+32}.2^2.D_6(5)$. The $C_G(f_-)$−module $[V, f_-]$ is a natural orthogonal 12−space for $\Omega^+_{12}(5)$ and the subspace X_1 is a singular 1-space. The quotient, $([V, f_-] + X_2)/X_2$ is 1−dimensional and $[V, f_-] \cap X_2 = X_1 \oplus [C_{X_2}(Z_1), Q_1]$.

(c) $C_G(f_-) \cap C_G(Z_1) \sim \langle f_- \rangle \times 2^2.D_6(5)$

(d) $C_V(f_-) \leq [X_3, Z_1] + [V, f_-] + X_2$

Proof: (a): Pick $s \in Z(L)$ and $e \in L'$ with $u = se$. Note that $e \in \tilde{F}$ and so $e^2 \in \tilde{F}' = F'$ and $e^2 \in Q_1$. In particular, $u^2F' = s^2F'$. Since $F/F' \cong C_4$ and $F = \langle u \rangle Z(F)F'$ we get $s^2 \neq 1$ and $|s| = 4$. Since s inverts Q, it is enough to show that e inverts f_-. Let $D = C_{L'}(Z_1)$. By Lemma 2.3, $C_Q(Z_1)$ has order 5^{11} and $\langle f_- \rangle$ is the unique cyclic subgroup of Q normalized by $\widetilde{P}_1$ and so by Lemma 2.5a, D has shape $4.D_5(5).4$, moreover D normalizes $\langle f_- \rangle$ and $C_D(f_-) = D'$. As a D−module, Q is the direct sum of irreducible modules of dimensions 1, 10 and 16. Using Lemma 2.3ab we conclude, by the action of $\widetilde{P}_1$ on the 10-dimensional space, that $\widetilde{P}_1$ is contained in a subgroup Y of D, such that Y/Z_1 is isomorphic to the subgroup of index 4 in $D_8 \wr Sym(5)$. In addition, $C_Y(f_+) \cap D' = Z_1$, and so $su \notin D'$. Thus e does not centralize f_-. As $e^2 \in Q_1$, the element e^2 does centralize f_- and so e inverts f_-. Thus (a) holds.

(b): As D lies in a parabolic P (of shape $5^{16}D$ in $E_6(5)$) that fixes a 1−space in Q, the group $\langle f_- \rangle$ is normalized by the parabolic $QZ(L)P$ of G and thus is a root group. Hence also the second, third and fourth statements in (b) hold. As Q centralizes V/X_3 and X_3/X_2 but not V/X_2 and as V/X_3 is one dimensional, $([V, f_-] + X_2)/X_2$ is 1-dimensional. Now $[X_2, f_-] = X_1$

and $([X_2, Q_1] + X_1)/X_1$ is the unique 10-dimensional subspace of X_2/X_1 invariant under P_1 and thus also the last statement in (b) holds.

(c): By (b), Z_1 centralizes $[V, f_-]$ and so (c) holds.

(d): As Z_1 centralizes $[V, f_-]$, the element f_- centralizes $[V, Z_-]$. Now both $C_V(f_-)+X_2/X_2$ and $[X_3, Z_1] + [V, f_-] + X_2/X_2$ are 17-dimensional and (d) holds. **QED**

Since $\tilde{S} = O_2(O^2(\tilde{P}_1))\langle u\rangle\langle r\rangle$, the group $\widetilde{S}$ normalizes $O^2(\widetilde{P}_1)\langle f\rangle$. Put $S = \widetilde{S}$, $P_1 = \langle S, f\rangle, P_2 = \widetilde{P}_2$ and $F = \langle P_1, P_2\rangle$.

Lemma 2.7 *(a) F normalizes a complement M_0 in X_3 to X_2. Put $M = M_0 + X_1$. Then $N_E(M) \cap Q = 1$. In particular, $F \cap Q = 1$ and F fulfills (+).*

(b) P_1 normalizes exactly two 1-spaces in V namely X_1 and U_1, where $U_1 \le M_0$. Moreover, $C_V(O^2(P_1)) = X_1 + U_1$.

(c) $[V, f_-] = X_1 + U_1 + [C_{X_2}(Z_1), Q_1]$

Proof: Put $U_1 = [C_V(Q_1), f, r]$. Note that Q_1 centralizes a 1-space in each of the modules X_i/X_{i-1}, and $C_V(Q_1)$ is 4-dimensional. As $[V, Q] \not\le X_2$, $[V, f_-] \not\le X_2$. Since $[X_3, f_-] \le X_2$ and $V = C_V(Q_1)X_3$, we conclude $[C_V(Q_1), f_-] \not\le X_2$. On the other hand as L acts completely reducibly on V, we have $[C_V(Q_1), f_+] = 1$. Thus $[C_V(Q_1), f] \not\le X_2$. By Lemma 2.3ad applied to F' and the modules X_3/X_2 and X_2/X_1, the element r^2 inverts $C_{X_3/X_1}(Q_1)$. In particular, $[r, f]$ centralizes X_3 and, as r inverts f and f has odd order, we conclude that f centralizes $C_{X_3/X_1}(Q_1)$. It follows that $([C_V(Q_1), f] + X_1)/X_1$ is $1-$dimensional and not contained in X_2/X_1. Hence U_1 is $1-$dimensional, $U_1 \le X_3$, $U_1 \not\le X_2$ and P_1 normalizes U_1.

Let $U_2 = \langle U_1^{P_2}\rangle$. As $|P_2/S| = 3$, the space U_2 is at most 3-dimensional. By Lemma 2.3 applied to $W = X_3/X_2$, we know that $(U_2 + X_2)/X_2$ is 3-dimensional. Thus U_2 is a 3-space and $U_2 \cap X_2 = 1$. Let $U_3 = \langle [U_2, Q_1]^{P_1}\rangle$. Since $[U_2, Q_1]$ is a 2-space and $|P_1/S| = 5$, we similarly have that U_3 is 10-dimensional and $U_3 \cap X_2 = 1$. Let $U_4 = \langle [U_3, Z_2]^{P_2}\rangle$. Since $[U_3, Z_2]$ is eight dimensional, U_4 is 24-dimensional and $U_4 \cap X_2 = 1$. Let $U_5 = [U_4, Z_1]$. Since X_3/X_2 and X_2/X_1 are dual as $E-$modules, Lemma 2.3af implies that $[V, Z_1]X_2/X_2$ and $[X_2, Z_1]$ are isomorphic and absolutely irreducible as P_1-modules and as $S-$modules. (Note here that Z_1 is trivial on X_1 and V/X_3.) By Lemma 2.3c, $[V, Z_1]$ splits over $[X_2, Z_1]$ as a P_1-module. It is now easy to see that every $S-$submodule of $[V, Z_1]$ is invariant under P_1. In particular P_1 normalizes U_5. Put $M_0 = U_2 + U_4$. Note that by Lemma 2.3, $(M_0 + X_2)/X_2 = X_3/X_2$ and that M_0 has dimension at most $3 + 24 = 27$. Thus $M_0 \cap X_2 = 0$ and M_0 is a complement to X_2 in X_3. Then

$$M_0 = U_1 + (U_2 \cap U_3) + (U_3 \cap U_4) + U_5 = U_1 + U_3 + U_5.$$

and so $F = \langle P_1, P_2\rangle$ normalizes M_0.

It follows that $[X_3, N_E(M) \cap Q] \le M \cap X_2 = X_1$. Since Q does not centralize X_3/X_1 and E is irreducible on Q, we have $N_E(M) \cap Q = 1$. Since $FQ = \widetilde{F}Q$, the last statement of (a) also holds.

As f_+ centralizes $C_V(Q_1)$ the last statement in (b) holds once we prove that $C_{C_V(Q_1)}(f_-) = X_1 + U_1$. As seen above $[C_V(Q_1), f_-] \not\le X_2$ and by a dual argument $[C_V(Q_1), f_-] \neq 0$. Hence $C_{C_V(Q_1)}(f_-)$ is at most 2-dimensional and so $C_{C_V(Q_1)}(f_-) = X_1 + U_1$. As r centralizes X_1 but not U_1, the P_1-modules X_1 and U_1 are not isomorphic. Thus (b) holds.

As seen above $U_1 \leq [V, f_-]$. Thus (c) follows from Lemma 2.6b. **QED**

Let $\chi_0 = X_1^G$ and $\zeta_0 = Z_1^G$.

Lemma 2.8 *$U_1 \in \chi_0$ and $C_M(Z_1) = \bigoplus \{X \mid X \in \chi_0,\ X \leq C_M(Z_1)\}$.*

Proof: We claim that E acts transitively on $\{Z \in \zeta_0 \mid [X_1, Z] = 1\}$. Indeed, for any such Z, we have $\dim[V, Z] = 32 = \dim[X_3/X_1, Z] = 2 \cdot \dim[X_2/X_1, Z]$. Thus the claim follows from Lemma 2.5a. By this claim, G acts transitively on $\{(Z, X) \in \zeta_0 \times \chi_0 \mid [X, Z] = 1\}$ and thus $C_G(Z_1)$ acts transitively on $\{X \in \chi_0 \mid [X, Z_1] = 1\}$. By Lemma 2.5b, $C_G(Z_1) \sim 2^2.(L_2(5) \times D_6(5)).2$ and $C_V(Z_1)$ is the tensor product of natural modules for $SL_2(5)$ and $\Omega_{12}^+(5)$. By Lemma 2.6c, f_- lies in the factor $SL_2(5)$ and so $[V, f_-]$ is one of six 12-spaces invariant under $2^2.D_6(5)$. By Lemma 2.6b, X_1 is a singular point in $[V, f_-]$ and it follows that $\{X \in \chi_0 \mid [X, Z_1] = 1\}$ is precisely the union of the sets of singular points in the six 12-spaces. By Lemma 2.7c, $X_1 + U_1 = C_{[V,f_-]}(Q_1)$ and so $X_1 + U_1$ is a non-degenerate subspace of $[V, f_-]$. As X_1 is singular in $[V, f_-]$, the space $X_1 + U_1$ is of "+"-type and so contains exactly one member of χ_0 distinct from X_1. By Lemma 2.5c, U_1 is the only $1-$space in $X_1 + U_1$, which is invariant under P_1 and distinct from X_1. So $U_1 \in \chi_0$. By the proof of Lemma 2.3ab, $[C_{M_0}(Z_1), V_1]$ is the direct sum of 5 pairwise non-isomorphic irreducible Q_1-modules, each of which is the direct sum of two conjugates of U_1 under F. Moreover, Q_1 normalizes each of the six 12-spaces and it is now easy to see that $C_M(Z_1)$ intersects each of the six 12-spaces in a 2-space of "+"- type. This clearly implies the lemma. **QED**

Lemma 2.9 *$N_E(M) \cap C_G(Z_1) = P_1$ and F contains a Sylow 2-subgroup of $N_E(M)$.*

Proof: Put $N = N_E(M)$ and $Y = C_N(Z_1)$. By Lemma 2.7, $N \cap Q = 1$. By Lemma 2.5a,

$$C_E(Z_1) \sim 5^{11}.(C_4 \times 4.D_5(5).4)$$

and $C_E(Z_1)$ normalizes a 10-space in X_3/X_2. By the proof of Lemma 2.8, Y normalizes $X_1 + U_1$ and a decomposition of this 10-space into an orthogonal sum of five 2-spaces of "+"-type. Let K be the full normalizer in $C_E(Z_1)$ of this decomposition. Then

$$K \sim 5^{11}(C_4 \times 2.\frac{1}{2}\,(D_8 \wr Sym(5)).2).$$

Let M_1 be the complement in X_3 to X_2 normalized by L. We claim that $U_3 \not\leq M_1$ and $U_5 \not\leq M_1$. As U_1 is $S-$invariant and $S \leq L$, we have $U_1 \leq M_1$. Since $P_2 \leq L$ we get $U_2 \leq M_1$. Suppose that $U_3 \leq M_1$. Then $\tilde{P}_1$ normalizes U_3 and since $f_- \in P_1\widetilde{P_1}$ we conclude that f_- normalizes U_3. Note that f_- centralizes X_3/X_2 and $U_3 \leq X_3$. Hence $[U_3, f_-] \leq X_2 \cap U_2 = 0$ and so f_- centralizes U_3, a contradiction to Lemma 2.6d. Thus U_3 is not contained in M_1. It follows that neither $U_3 \cap U_4$ nor U_4 are contained in M_1. As $P_2 \leq L$ acts irreducibly on U_4 we conclude that $U_4 \cap M_1 = 0$ and so finally $U_5 \not\leq M_1$.

As $Y \cap Q = 1$, Y is isomorphic to a subgroup of $K/O_3(K)$ and so $Y/O_2(Y)$ is isomorphic to a section of $Sym(5)$. As 5 divides the order of P_1 we conclude that $O_2(P_1O_2(Y)/O_2(Y)) = 1$ and so $O_2(P_1) \leq O_2(Y)$. As $C_Q(O_2(P_1)) = \langle f_- \rangle$, any conjugate of L in E containing $O_2(P_1)$ is of the form L^x for some $x \in \langle f_- \rangle$. By Sylow's theorem $O_2(Y)$ lies in some conjugate of L in

E and so $O_2(Y) \leq L^x$ for some $x \in \langle f_- \rangle$. Thus $O_2(Y)$ normalizes $[M_1, Z_1]^x$. By Lemma 2.6b, x centralizes $[M_1, Z_1]$ and so $O_2(Y)$ normalizes $[M_1, Z_1]$, $[X_2, Z_1]$ and $U_5 = [M, Z_1]$. Since all of these $Y-$modules are irreducible and $U_5 \leq [V, Z_1] = [X_2, Z_1] \oplus [M_1, Z_1]$ we conclude that these $O_2(Y)-$modules are pairwise isomorphic and so $O_2(Y)$ acts self-dually on $[X_2, Z_1]$.

For each $X \leq E$, we write X^* for $C_X(X_1)$. Since $Z(L)$ acts transitively on $X_1^{\#}$, the groups $\tilde{F}, F$ and P_1 also act transitively on $X_1^{\#}$. Thus $Y = Y^* P_1$. Let C (respectively $\hat{C}$) be the largest subgroup of K^* centralizing (centralizing or inverting) $(U_3 + X_2)/X_2$. Then $\hat{C}Q/Q \cong C_4$. Since $O_2(Y)$ acts self-dually on $[X_2, Z_1]$ and the elements of $\hat{C}$ act as scalars we conclude that $\hat{C} \cap Y$ centralizes or inverts $[X_2, Z_1]$. Hence $\hat{C} \cap Y^* \leq C$ and $Y^*C \cap \hat{C} = (Y^* \cap \hat{C})C = C$. Thus no element of Y^*C/C inverts $(U_3 + X_2)/X_2$. We are now in a position to apply Lemma 2.4 (with $K\langle t \rangle = K^*/C$ and $A = O_2(Y^*)C/C \cap O_2(O^2(K^*/C))$. Note that A is not abelian as $Q_1C/C \leq A$ and $Q_1 \cap C = Z_1$. As $r \in P_1^*$, the group P_1^* has $Frob_{20}$ as a quotient. Since Y^*C normalizes A we conclude that $Y^*C \leq O_2(Y^*)P_1^*$ and $A = Q_1C/C$. From the structure of D_8, from the last two statements in Lemma 2.4 and from $O_2(Y^*) \cap \hat{C} \leq C$ we conclude that $O_2(Y^*)C = Q_1C$. Thus $Y^*C = P_1^*C$. Since $C \cap Y^* \cap Q = 1$, we have $C \cap Y^* = Z_1 \leq P_1^*$. So $Y^* = P_1^*$ and $Y = P_1$, proving the first statement of the lemma.

Since $Z_1 = Z(S) \cap S'$, $N_G(S) \leq C_G(Z_1)$ and so S a Sylow 2-subgroup of N. **QED**.

Lemma 2.10 *There exists $t \in N_G(P_1) \cap N_G(M)$ with $X_1 \neq X_1^t$.*

Proof: Let $t \in N_G(P_1)$. By Lemma 2.7b, $C_V(O^2(P_1)) = U_1 + X_1$ and so t normalizes $U_1 + X_1$. Also by Lemma 2.3ae, the isomorphism types of the two $10-$dimensional modules in V invariant under P_1 are not conjugate under an automorphism of P_1. So t normalizes U_3. Hence $t \in N_G(M)$ if and only if $t \in N_G(U_5)$.

Note that P_1 is contained in $\langle f_- \rangle\, C_L(Z_1)$. Let $\langle t_1 \rangle = Z(C_{L'}(Z_1))$. Then by Lemma 2.5a, t_1 is of order 4 and $[t_1, P_1] \leq Z_1$. So $t_1 \in N_G(P_1)$. Further $C_G(Z_1)$ is of shape $2^2.(L_2(q) \times D_6(5)).2$. Note that $[V, f_-]$ is 12-dimensional and there exists a quadratic form on $[V, f_-]$ invariant (up to scalar multiplication) under the action of $N_G(\langle f_- \rangle)$. Moreover, $[V, f_-]$ is equal to the sum of X_1, U_1 and the 10-dimensional subspace of X_2, which is normalized by P_1. In particular P_1 normalizes a decomposition of $[V, f_-]$ into an orthogonal sum of six 2-dimensional subspaces of "+"-type. Let T_0 be the largest subgroup of $2^2.D_6(5)$, which normalizes this decomposition and normalizes $X_1 + U_1$. Then P_1 normalizes T_0 and T_0/Z_1 is isomorphic to a subgroup of index 4 in $(D_8 \times D_8 \wr \Sigma_5)$.

Let $T = C_{T_0}(f)$. Then $T/Z_1 \cong D_8 \times C_2$ and T normalizes $O^2(P_1)$ and so also Q_1 and $V_1 = Q_1'$. Moreover, $t_1 \in T$, $[t_1, T] \leq Z_1$, $T' = Z(G)Z_1$ and $T \cap P_1/Z_1 \cong C_4$. Pick $t_2 \in T$ with $t_2^2 \in Z_1$ and $[t_2, T] \not\leq Z_1$. We will show that either t_2 or t_1t_2 fulfills the conclusion of the lemma. It is easy to check in T_0 that t_2 centralizes a 1-space in each of the six 2-spaces and that these six 1-spaces form a 6-space of "+"-type. As the central involution of $\Omega_6^+(5)$ lifts to an element of order four in $2.\Omega_6^+(5) \cong SL_4(5)$ we conclude that t_2 and by symmetry t_1t_2 are elements of order 4. It follows that

$$\langle t_1, t_2 \rangle \cong Q_8 \quad \text{and} \quad \langle t_1, t_2 \rangle \cap P_1 = Z_1.$$

We will now examine the action of $\langle t_1, t_2 \rangle$ on the set Π of 16-spaces in V invariant under $O^2(P_1)$. Note that $|\Pi| = 5 + 1 = 6$. Put $H = [V_1, f]$. Then $\langle t_1, t_2 \rangle$ normalizes H and acts faithfully on the 2-dimensional space $C_{[V,Z_1]}(H)$. So the orbits of $\langle t_1, t_2 \rangle$ on the 1-dimensional subspaces of $C_{[V,Z_1]}(H)$, and hence also on Π, are all of length 2. Now by Lemma 2.9, t_1 does not normalize M and so $[M, Z_1] \neq [M, Z_1]^{t_1}$. Hence $[M, Z_1] = [M, Z_1]^t$ for $t = t_2$ or for $t = t_1t_2$. Recall that $U_5 = [M, Z_1]$ and thus the selected t is in $N_G(M)$.

It remains to show that t normalizes P_1. Since t normalizes $O^2(P_1)$, it is enough to prove that $[t_1, N_S(\langle f\rangle)] \leq P_1$. Since $T = \langle t_1, t_2\rangle (T \cap P_1)$, $T \cap P_1$ is the largest subgroup of T acting trivially on Π. As $N_S(\langle f\rangle)$ acts trivially on Π, the same is true for $[t_1, N_S(\langle f\rangle)]$ and so $[t_1, N_S(\langle f\rangle)] \leq T \cap P_1 \leq P_1$, completing the proof of the Lemma. **QED**

Lemma 2.11 *Let $R = \langle F, F^t\rangle$. Then F has three orbits on the right cosets of F in R. The orbit stabilizers are F, P_1 and a group of order $2^5{\cdot}3{\cdot}5^2{\cdot}13$. In particular $|R| = 2^{15}{\cdot}3^3{\cdot}5^3{\cdot}7{\cdot}13{\cdot}29$.*

Proof: Note first that R normalizes M. We will divide the proof into several steps.

1 *Let $\chi = X_1^R$ and $\zeta = z_1^R$. Then R acts transitively on*
(a) $\{(X, z) \mid X \in \chi,\ z \in \zeta$ *and* z *centralizes* $X\}$,
(b) $\{(X, z) \mid X \in \chi, z \in \zeta$ *and* z *inverts* $X\}$.

Clearly a Sylow 2-subgroup of $N_R(X_1)$ contains representatives of each class of involutions in $N_R(X_1)$. By Lemma 2.9, S is Sylow 2−subgroup of $N_R(X_1)$ and so F contains representatives of each class of involutions in $N_R(X_1)$. As $F/F' \cong C_4$, all the involutions in F are contained in $Z(G) \times F'$. By Lemma 2.3, F' has two classes of involutions with representatives z_1 and i. Moreover $C_W(i)$ is 15-dimensional and so $\dim C_V(i) = 2 + 2 \cdot 15 = 32$. Hence F has two orbits on $F \cap \zeta$ with representatives z_1 and z_0i, where z_0 is the central involution in G. Now z_1 centralizes X_1 and z_0i inverts X_1. Thus (1) holds.

2 *(a)* $C_R(z_1)$ *acts transitively on* $C_M(z_1) \cap \chi$ *and* $|C_M(z_1) \cap \chi| = 12$,
(b) $C_R(z_1)$ *and* P_1 *act transitively on* $[M, z_1] \cap \chi$ *and* $|[M, z_1] \cap \chi| = 80$,
(c) $|C_R(z_1)| = 2^{15} \cdot 3 \cdot 5$,
(d) $C_{N_R(X_1)}(i) \leq F$.

The two transitivity statements for $C_R(z_1)$ follow from (1). By Lemma 2.8, $|C_M(z_1) \cap \chi_0| = 12$. Moreover, by Lemma 2.3, U_3 contains 10 elements of U_1^F and so $C_M(z_1) \cap \chi$ contains at least the 11 elements of $C_M(z_1) \cap \chi_0 \setminus \{X_1\}$. Conjugation by t shows that $C_M(z_1) \cap \chi$ also contains the 11 elements of $C_M(z_1) \cap \chi_0 \setminus \{U_1\}$. Thus $C_M(z_1) \cap \chi = C_M(z_1) \cap \chi_0$ and (a) holds. By Lemma 2.9, $C_R(z_1) \cap N_R(X_1) = P_1$ and so

$$|C_R(z_1)| = 12 \cdot |C_R(z_1) \cap N_R(X_1)| = 12 \cdot |P_1| = 2^{15} \cdot 3 \cdot 5.$$

Now $|[M, z_1] \cap \chi| = |C_R(z_1)| \,/\, |C_R(z_1) \cap N_R(X)|$, where $X \in \chi$ is inverted by z_1. By Lemma 2.3d, $|C_F(i)| = 2^{11} \cdot 3$. Further, $|C_R(z_1) \cap N_R(X)| \geq |C_F(i)|$ and so $|[M, z_1] \cap \chi| \leq 80$. Finally, P_1 has an orbit of length 80 on $[M, z_1] \cap \chi$. Indeed, there are 80 points at distance 4 from a in Γ_0 (the generalized octagon associated to F), these 80 points correspond to 80 elements in U_0^F and U_0^F is a subset of χ. So $|[M, z_1] \cap \chi| = 80$, moreover $|C_R(z_1) \cap N_R(X)| = |C_F(i)|$ and P_1 acts transitively on $[M, z_1] \cap \chi$. This completes the proof of (2).

3 $N_R(X_1) = F = N_R(F)$. *In particular, the actions of R on X_1^R and on R/F are isomorphic.*

Let $N = C_R(X_1)$. Then, as in (1), N has two classes of involutions, with representatives z_1 and i. By (2)(d) and Lemma 2.8, F' contains the centralizers of z_1 and i in N. Hence by a standard argument, see for example [6, 9.2.1], $F' = N$. Thus $N_R(X_1) = F$. Now X_1 is the unique 1-space in M normalized by F and so $N_R(F) \leq N_R(X_1)$.

Let Γ be the graph with vertices χ and edges $\{X_1, U_1\}^R$. Since $R = \langle F, F^t \rangle$ we have

4 Γ *is connected.*

5 *Let $a, b \in \chi$. Suppose there exists $z \in \zeta$, such that z normalizes a and b. Then a and b have distance at most 2 in Γ. If z centralizes a or b, then a and b have distance at most 1.*

Suppose first that z centralizes a. Then we may assume without loss that $a = X_1$ and $z = z_1$. Then (2) implies that $b = X_1$ or $b \in U_1^F$ and so a and b are at distance at most 1.

In the general case pick $c \in \chi$ so that z centralizes c. Then a and b are at distance at most 1 from c, and (5) is proved.

For a vertex a, put $R_a = N_R(a)$, and for an edge $\{a, b\}$, let $1 \neq z(a,b) \in Z(R'_a \cap R'_b)$. Note that, if $(a,b) = (X_1, U_1)^g$, then $z(a,b) = z_1^g = z(b,a)$. For $g \in F$, we identify $\alpha^g \in \Gamma_0$ with $U_1^g \in \Gamma$.

6 *(a) R acts transitively on geodesics of length 2 in Γ. Moreover, the stabilizer of a geodesic of length 2 is isomorphic to $C_2 \times Frob_{20}$*

(b) Let d and e be at distance 2 in Γ. Then $R_d \cap R_e$ acts transitively on the set of pairs (a,b) such that $\{a,b\}$ is an edge with $z(a,b) \in R_d \cap R_e$. Moreover, $R_a \cap R_b \cap R_d \cap R_e$ is isomorphic to $C_2 \times C_4$.

Let a and b be in α^F. Suppose that a and b are at distance less than or equal to 6 in Γ_0. Then there exists $c \in \alpha^F$, such that c is (in Γ_0) at distance 2 from a and at distance at most 4 from b. Put $z = z(X_1, c)$. Then by Lemma 2.3ab&ac, z centralizes a and normalizes b. Thus (5) implies that either $a = b$ or a is adjacent to b in Γ. Suppose that every pair of elements in α^F are adjacent in Γ. Then every pair of elements in $\alpha^F \cup \{X_1\}$ are adjacent. Since Γ is connected, we conclude that $\alpha^F \cup \{X_1\}$ is the set of vertices of Γ. Hence $|R| = |F| \cdot (|F|+1)$ and so $|R|_2 = 2^{14}$, a contradiction to (2)(c). So there are two elements of α^F, that have distance 8 in Γ_0, and have distance 2 in Γ. Since P_1 is transitive on $\triangle^8(\alpha)$ and since every geodesic of length two in Γ is conjugate to one with X_1 as its midpoint, we conclude that R is transitive on geodesics of length 2 in Γ. Moreover, the stabilizer in F of two elements of distance 8 in α^F is a $C_2 \times F_{20}$. Hence (a) is proved.

To prove (b) we assume without loss that $a = X_1$ and $b = U_1$. Since $z(a,b)$ normalizes d and e, we get by (5) that $d, e \in \alpha^F$ and that d and e are at distance less than or equal to 4 from b in Γ_0. Since d and e are at distance 8 from each other, b lies on a geodesic from d to e in Γ_0 and is at distance 4 from both d and e. Now F acts transitively on paths of length 8 in Γ_0 starting with a vertex in α^F and the stabilizer of such a path is a $C_2 \times C_4$. This proves (b).

7 *Let d and e be at distance 2 in Γ. Then $R_d \cap R_e$ acts transitively on $R_d \cap R_e \cap \zeta$. Moreover, if $z \in R_d \cap R_e \cap \zeta$, then $R_d \cap R_e \cap C_R(z)$ has order $2^5.3$ and has a normal Sylow $3-$subgroup.*

Let $z \in R_d \cap R_e \cap \zeta$. By (6), $R_d \cap R_e$ acts transitively on $R_d \cap R_e \cap \zeta$ and $R_d \cap R_e \cap C_R(z)$ acts transitively on all pairs (a,b) such that $\{a,b\}$ is an edge with $z = z(a,b)$. Put $A = R_a \cap R_b \cap R_d \cap R_e$ and $B = R_d \cap R_e \cap C_R(z)$. Then $|A| = 8$. Next we show that there are exactly 12 choices for (a,b). Indeed a is in $C_V(z) \cap \chi$ and so by (2)(a), there are exactly twelve choices for a. Moreover, as $R_a \cap R_b = C_{R_a}(z)$, b is uniquely determine by z and a. It follows that $|B| = 8 \cdot 12 = 2^5 \cdot 3$. We claim that A is normal in B. For this let $\{\bar{a}, \bar{b}\}$ be an edge different from $\{a,b\}$ with $z = z(\bar{a},\bar{b})$. Again choose notation so that $a = X_1$. Since $z = z(\bar{a},\bar{b})$, z centralizes $\bar{a}$ and $\bar{b}$ and so $\bar{a}$ and $\bar{b}$ are at distance 2 from b in Γ_0. It follows from [5] that A normalizes $\bar{a}$ and $\bar{b}$, and so $A = R_{\bar{a}} \cap R_{\bar{b}} \cap R_d \cap R_e$. Thus A is independent of the choice of (a,b) and therefore normal in B. Let $\widetilde{B} = C_B(d)$ and $\widetilde{A} = C_A(d)$. Let $g \in A$ with $|g| = 4$. As $\langle g \rangle$ acts faithfully on the group of order five in $R_a \cap R_d \cap R_e$, $g^2 \neq z_0$. As $g \notin Z(G)F'$, $g^2 \notin F'$ and so $g^2 \neq z$. Thus $g^2 = z_0 z$. Since z and z_0 invert d, g^2 centralizes d. Thus $A/\widetilde{A}$ is elementary abelian, $|A/\widetilde{A}| = 2$ and $\widetilde{A} \cong C_4$. Let i be the involution in $\widetilde{A}$. Suppose that the Sylow 3$-$subgroups of B are not normal in B. Since 3 divides $\widetilde{B}$ and $|\widetilde{B}/\widetilde{A}|_2 \leq 4$, $\widetilde{B}/\widetilde{A} \cong A_4$. Let $Y = O_2(\widetilde{B})$ and $Y^* = [Y, O^2(\widetilde{B})]$. Then $Y = Y^*\widetilde{A}$ and $[Y^*, \widetilde{A}] = 1$. Hence $\widetilde{A} \leq Z(Y)$ and $Y/Z(Y)$ is elementary abelian. It follows that $|Y'| \leq 2$. Hence either Y is abelian, $Y^* \cap \widetilde{A} = 1$ and $Y \cong C_4 \times C_2 \times C_2$ or Y is not abelian, $Y^* \cong Q_8$ and $Y \cong C_4 \circ Q_8 \cong C_4 \circ D_8$. In particular, $\langle i \rangle = \Phi(Y)$. Note that $\widetilde{A} = C_R(d) \cap R_e \cap C_R(i)$. We claim that all involutions in $C_R(d) \cap R_e$ are conjugate under $R_d \cap R_e$. Indeed let j be any such involution. Since d and e have distance 2, (5) implies that $j \notin \zeta$. Thus $z_0 j \in \zeta$ and by the first part of (7), the conjugacy class of $z_0 j$ in $R_d \cap R_e$ is uniquely determined. Thus the claim holds. Moreover, by the structure of Y there exists an involution j in $\widetilde{B}$ different from i. Then by the claim $j = i^h$ for some $h \in R_d \cap R_e$. Now $[\widetilde{A}, j] = 1$ and $\widetilde{A} \leq C_R(d) \cap R_e \cap C_R(j) = \widetilde{B}^h$. As $\widetilde{B}^h/Y^h$ has order three, $\widetilde{A} \leq Y^h$. Hence

$$i \in \Phi(\widetilde{A}) \leq \Phi(Y^h) = \langle i^h \rangle = \langle j \rangle$$

a contradiction, which proves (7).

8 *Let a be at distance 2 from X_1 in Γ. Then $|F \cap R_a| = 2^5 \cdot 3 \cdot 5^2 \cdot 13$ and $F \cap R_a$ has exactly two orbits on the neighbors of X_1 in Γ.*

Let $z \in F \cap R_a \cap \zeta$ and D be the Sylow 3$-$subgroup of $C_{F \cap R_a}(z)$. By Lemma 2.3e $|N_F(D)| = 3^3 \cdot 2^5$ and so by (7), $C_{F \cap R_a}(z)$ contains a Sylow 2-subgroup of $N_F(D)$. Put $K = F' \cap R_a$. Then by Lemma 2.3, $C_K(z) \cong D_{24}$. Since $F \cap R_a$ acts transitively on the involutions in K, we conclude that the Sylow 2-subgroups of K are dihedral groups of order 8 and that K has exactly one class of involutions. By (6), $|K|$ is divisible by 5, and since $|F'| = 2^{11} \cdot 3^3 \cdot 5^2 \cdot 13$, we have $|K| = 2^3 \cdot 3^{1+u} \cdot 5^{1+v} \cdot 13^w$, where u is 0, 1 or 2, and v and w are 0 or 1. Since $C_K(z)$ is a maximal subgroup of $N_{F'}(D)$, u is 0 or 2. We claim that K has an orbit on α^F with orbit stabilizer $C_2 \times Frob_{20}$ and an orbit with an orbit stabilizer of order 2^5. Indeed let b be at distance 1 from X_1 and a in Γ. Then by (6), $F \cap R_b \cap R_a \cong C_2 \times Frob_{20}$. Moreover, any 2-subgroup of F fixes a point in α^F and so there exists c in α^F so that 2^5 divides $|F \cap R_a \cap R_c|$. Suppose 5 divides $|F \cap R_a \cap R_c|$. Then $2 \leq |O_2(F' \cap R_a \cap R_c)| \leq 8$ and so $O^2(F' \cap R_a \cap R_c)$ centralizes $O_2(F' \cap R_a \cap R_c)$, a contradiction since the involutions in K are not centralized by elements of order 5 in K. So $|F \cap R_a \cap R_c| = 2^5$. In particular

$$1755 = |\alpha^F| \geq |b^K| + |c^K| = 2^2 \cdot 3^{1+u} \cdot 5^v \cdot 13^w + 3^{1+u} \cdot 5^{1+v} \cdot 13^w,$$

and so

$$(*) \qquad 65 \geq 3^u \cdot 5^v \cdot 13^w.$$

Now $|K/N_K(D)| = 5^{1+v} \cdot 13^w$ and $|K/N_K(D)|$ is congruent to 1 modulo 3, so we must have $v = 1$. By Lemma 2.3, the centralizers of elements of order 5 in F are $\{2,5\}-$groups and no involution in K is centralized by an element of order five . Thus the centralizers of elements of order five in K are 5-groups. In particular the Sylow 5-subgroups of K are $TI-$sets, and so the number of Sylow 5-subgroups in K is congruent to 1 modulo 25. Since no divisor of $2^3 \cdot 3^3$ is 1 mod 25, $w \neq 0$. Thus $w = 1$ and by $(*)$, $u = 0$ and the equal sign holds in $(*)$. This means that $F \cap R_b$ has no further orbit on α^F and (8) is proved.

We remark that using the list of maximal subgroups of the Tits group or the classification of groups with dihedral Sylow 2-subgroups it is not difficult to see that $K \cong L_2(25)$, but we will not need this fact.

9 *F has three orbits on R/F with lengths 1, 1755 and 2304.*

In view of (8) it is enough to prove that there exist no points at distance 3 from X_1 in Γ. One easily checks in Γ_0 that there exist points b, c, d in α^F such that b has distance 8 from both c and d in Γ_0 and c and d are at distance 2 in Γ_0. Then c and d are adjacent in Γ and b is at distance 2 from c and d in Γ. Let a be at distance 2 from X_1. By (8), $F \cap R_a$ has two orbits on the neighbors of a in Γ. One orbit is the set of common neighbors of X_1 and a. By (6) there exists $g \in R$ with $b^g = X_1$ and $c^g = a$. Then d^g lies in the second orbit and has distance 2 from X_1 In particular every point adjacent to a is at distance at most 2 to X_1 in Γ. This completes the proof of (9) and of Lemma 2.11. **QED**

It now follows from Lemma 2.11 that R has the following properties:

(a) R has a subgroup F with $F/Z(G) \cong {}^2F_4(2)$.

(b) F has 3 orbits on R/F with lengths 1, 1755 and 2304.

By [12] we conclude that $\overline{R}$ is isomorphic to the Rudvalis group. This completes the proof of Theorem 2.1.

3 A Computer-free Construction of the Higman-Sims Group as a Subgroup of E$_7$(5).

In this chapter we will prove the following Theorem:

Theorem 3.1 *$E_7(5)$ contains subgroups isomorphic to M_{22} and the Higman-Sims group.*

We start with some of the properties of M_{22} we will need in the proof of the theorem.

Lemma 3.2 *Let $M = M_{22}$ act faithfully on $\Omega = \{1, 2, \ldots, 22\}$, let $\omega \in \Omega$ and let D be the stabilizer of a hexad $\mathcal{H}$ in Ω.*

(a) M_ω *acts transitively on* $\Omega \setminus \{\omega\}$ *and* $M_\omega \cong L_3(4)$.

(b) $D \sim 2^4.Alt(6)$, D *acts transitively on* $\Omega \setminus \mathcal{H}$ *and any subgroup of shape* $2^4Alt(6)$ *in* M *is conjugate to* D.

(c) *Let* $E \leq M_\omega$ *with* $E \cong 2^4Alt(5)$. *Then* E *has orbits of lengths 1, 5 and 16 or of lengths 1, 1 and 20 on* Ω. *In the first case* E *stabilizes a hexad, and in the second case* $N_M(E) \sim 2^4Sym(5)$.

(d) *Let* $A \leq B \leq M$ *with* $A \cong Alt(6)$ *and* $B \cong Alt(7)$. *Then* A *is not contained in a conjugate of* D.

(e) *Let* $A \leq D$ *such that* $A \cong Alt(5)$ *and such that* $O_2(D)$ *is a natural* $SL_2(4)-$*module for* A. *Then* A *has orbits of lengths 1 and 5 on* $\mathcal{H}$ *and orbits of lengths 1 and 15 or of lengths 6 and 10 on* $\Omega \setminus \mathcal{H}$.

(f) *Let* $A \leq D$ *such that* $A \cong Alt(5)$ *and such that* $O_2(D)$ *is a natural* $\Omega_4{}^-(2)-$*module for* A. *Then* A *acts transitively on* $\mathcal{H}$ *and has orbits of lengths* $1, 5$ *and* 10 *on* $\Omega \setminus \mathcal{H}$.

(g) *If* $A \leq D$ *with* $A \cong Alt(6)$, *then* A *has orbits of lengths 1 and 15 or of lengths 6 and 10 on* $\Omega \setminus \mathcal{H}$.

(h) *If* $A \leq M_\omega$ *with* $A \cong Alt(6)$, *then* A *has orbits of lengths 1, 6 and 15 on* Ω.

(i) M *has no subgroup of index 56.*

Proof: The maximal subgroups of M_{22} and their orbits on Ω are listed in Table 10.3 on page 285 of [4]. We use this table without further reference. In particular, (a) and (b) hold. ¿From the definition of a Steiner system, the set $\Omega \setminus \{\omega\}$ together with the set of hexads containing ω form a projective plane of order 4. In particular both the two point stabilizer and the stabilizer of an incident point hexad pair have shape $2^4SL_2(4) = 2^4Alt(5)$ where the 2^4 is a natural $SL_2(4)-$module. Let E^* be the normalizer of a pair of points. Then $E^* \cong 2^4Sym(5)$ and (c) holds.

To prove (h), let f be an element of order five in A. Then f has exactly one fixed point η on $\Omega \setminus \{\omega\}$. Clearly A does not fix η and $|\eta^A|$ divides $|A|$ and is congruent to 1 modulo 5. Thus $|\eta^A| = 6$. As f acts fixed-point freely on the remaining 15 points and A has no orbit of length 5, (h) holds.

(g): By (h) we may assume that A has no fixed points on Ω. Then by the same argument as in the proof of (h), A has orbits of length 6 and 10 on $\Omega \setminus \mathcal{H}$.

For (e) and (f) we note that if A fixes a point η outside $\mathcal{H}$, then the action of A on $\Omega \setminus (\mathcal{H} \cup \{\omega\})$ is isomorphic to the action of A on $O_2(D)^\#$. Since in case (f) there exists a unique class of subgroups $Alt(5)$ in $O_2(D)A$, (f) holds. In case (e), we need to rule out the possibility that the orbits of A on $\Omega \setminus \mathcal{H}$ have lengths 5, 5 and 6. In this case, the elements of order three in A would have 4 fixed-points on $\Omega \setminus \mathcal{H}$, but only one fixed-point on $O_2(D)$, a contradiction.

(d) Note that B is unique up to conjugation and has an orbit Ξ of length 7. Hence A has orbits of length 1 and 6 on Ξ and so is transitive on $\Omega \setminus \Xi$ by (h). As no two hexads can intersect in a set of size 5, the orbit of length 6 is not a hexad. Thus (d) holds.

Since M has no maximal subgroup of index dividing 56, (i) holds. **QED**

For the convenience of the reader we recall the definition of the HS group as found [8]. Let (S, B) be the Steiner System of type $(3, 6, 22)$. Let $\mathcal{G}$ be the (undirected) graph with vertex set $\{*\} \cup S \cup \mathcal{B}$, where $*$ is a new symbol. In $\mathcal{G}$

(a) The vertex $*$ is joined to each point in S.

(b) Each point $\alpha \in S$ is joined to the 21 hexads containing α.

(c) Two hexads are joined if and only if they are disjoint.

Then by [8], $Aut(\mathcal{G})$ is transitive on $\mathcal{G}$ and $Aut(\mathcal{G})$ has a simple subgroup of index 2 and order 44,352,000, now called the Higman Sims group HS. We refer to $\mathcal{G}$ as the Higman-Sims graph.

The next two lemmas characterize M_{22} and HS in terms of certain subgroups.

Lemma 3.3 *Let M be a group and L, M_1 and M_2 subgroups of M such that $L \cong L_3(4)$, $M_1 \sim 2^4 Sym(5)$, $M_2 \sim 2^4 Alt(6)$, $L \cap M_1 \cong L \cap M_2 \sim 2^4 Alt(5)$, $M_1 \cap M_2 \sim 2^4 Sym(4)$, $L \cap M_1 \cap M_2 \sim 2^4 Alt(4)$ and $M = \langle L, M_1, M_2 \rangle$. Then $M \cong M_{22}$.*

Proof: Put $\Gamma = M/L$ and $\alpha = L \in \Gamma$. Let $t \in M_1 \cap M_2 \setminus L$. Since $M_1 \cap L$ is normal in M_1, we have $M_1 \cap L \le L^t$. As $M_1 \cap M_2$ does not normalize $L \cap M_2$, the element t does not normalize L and so $L \cap L^t = L \cap M_1$. Put $\Gamma_0 = \{\alpha\} \cup \alpha^{tL}$. Then $|\Gamma_0| = 1 + 21 = 22$.

Now $|\alpha^{M_2}| = |M_2/M_2 \cap L| = 6$ and $\alpha^{M_2} = \{\alpha\} \cup \alpha^{t(M_2 \cap L)} \subset \Gamma_0$. Further, as $M_1 = \langle t \rangle (M_1 \cap L)$, we have

$$\alpha^{M_2 M_1} = \alpha^{M_2 \langle t \rangle (M_1 \cap L)} = \alpha^{M_2 (M_1 \cap L)} = \alpha \cup \alpha^{t(M_2 \cap L)(M_1 \cap L)}.$$

Since $M_1 \cap L$ acts transitively on $\alpha^{tL} \setminus \{\alpha\}$ we get that $\alpha^{M_2 M_1} = \Gamma_0$. Hence M_1 and L normalize Γ_0. Note that $M_2 = \langle M_2 \cap L,\ M_2 \cap M_1 \rangle$ and so $M = \langle M_1,\ M_2,\ L \rangle = \langle M_1,\ L \rangle$. Thus M normalizes Γ_0, also $\Gamma = \Gamma_0$ and $|M/L| = 22$. Put $B = \alpha^{M_2}$ and $\mathcal{B} = \{B^m | m \in M\}$. We claim that $(\Gamma, \mathcal{B})$ is a Steiner System of type $(3, 6, 22)$. Since L is doubly transitive on $\Gamma \setminus \{\alpha\}$, M is triply transitive on Γ. Hence each set of three elements in Γ lies in e elements of $\mathcal{B}$ where e is a positive integer independent of the set of three. Counting tuples (H, a, b, c) such that $H \in \mathcal{B}$ and a, b, c are pairwise different elements of H we get

$$|\mathcal{B}| \cdot 6 \cdot 5 \cdot 4 = 22 \cdot 21 \cdot 20 \cdot e.$$

As $|\mathcal{B}| = |M/M_2| = 22 \cdot |L|/|M_2| = 77$ we get $e = 1$ and the claim is established.

Since $M \le Aut(\Gamma, \mathcal{B}) \cong Aut(M_{22})$ and $|M| = 22 \cdot |L| = |M_{22}|$, we deduce that $M \cong M_{22}$. **QED**

Lemma 3.4 *Let H be a group, M and D subgroups of H, and L a subgroup of M. Suppose that each of the following holds:*

(i) $M \cong M_{22}$, $L \cong L_3(4)$ and $D \sim 2^4 Sym(6)$,

(ii) There exists $t \in N_H(L)$ with $t^2 \in L$ such that $D \cap M \sim 2^4 Sym(5)$, $D \cap M^t \sim 2^4 Alt(6)$, and $D \cap D^t \sim 2^4 Sym(4)$.

(iii) $H = \langle M, L, D, t\rangle$.

Then H is isomorphic to HS, $C_2 \times HS$ or $Aut(HS)$.

Proof: Let Γ be the graph whose vertices are the right cosets of M in H, and whose edges are the sets $\{Mh, Mth\}$ for $h \in H$. Put $\alpha = M$ and $\beta = Mt$.

Note that $\langle H_\alpha, H_{\{\alpha,\beta\}}\rangle = \langle M, L\langle t\rangle\rangle$ and $D = \langle D \cap M,\ D \cap M^t\rangle$. So $\langle H_\alpha, H_{\{\alpha,\beta\}}\rangle = H$ and Γ is connected.

Let $\triangle^i(\alpha)$ be the set of vertices at distance exactly i from α and put $\triangle(\alpha) = \triangle^1(\alpha)$. Then $|\triangle(\alpha)| = 22$, $L = H_{\alpha\beta}$ and L acts transitively on $\triangle(\beta) \setminus \{\alpha\}$. Let $r \in D \cap M \setminus L$ and put $\gamma = \alpha^{trt}$. Then $\{\beta,\gamma\} = \{\alpha,\beta\}^{rt}$ and $\{\beta,\gamma\}$ is an edge. Since $r \in D$ and $D \cap M^t$ is normal in D, r normalizes $D \cap M^t$. Moreover, $t^2 \in L \leq M$ and so r^t normalizes $D^t \cap M$. Thus $D^t \cap M \leq H_\alpha \cap {H_\alpha}^{r^t} = H_{\alpha\gamma}$. In particular γ is not adjacent to α and thus $\gamma \in \triangle^2(\alpha)$.

Suppose that $H_{\alpha\gamma} \neq D^t \cap M$. As $D^t \cap M$ is maximal in M we get $H_{\alpha\gamma} = H_\alpha = H_\gamma$, and $\triangle(\alpha) \cap \triangle(\gamma) = \beta^{H\alpha} = \triangle(\alpha) = \triangle(\gamma)$, and since Γ is connected, $\Gamma = \{\alpha,\gamma\} \cup \triangle(\alpha)$. Thus L fixes exactly the vertices α, β and γ. Hence $\gamma = \gamma^t$ and $t \in H_\gamma = H_\alpha$, a contradiction. We have proved

1 *H acts transitively on geodesics of length 2, $H_{\alpha\gamma} \sim 2^4 Alt(6)$, $|\triangle^2(\alpha)| = 77$, $|\triangle(\alpha) \cap \triangle(\gamma)| = 6$ and $H_{\alpha\beta\gamma} \sim 2^4 Alt(5)$.*

By part (c) of Lemma 3.2, $H_{\alpha\beta\gamma}$ acts transitively on $\triangle(\gamma)\setminus\triangle(\alpha)$. Let $\delta \in \triangle(\gamma)\setminus\triangle(\alpha)$. Then $H_{\alpha\gamma\delta} \cong Alt(6)$ and $H_{\alpha\beta\gamma\delta} \cong Alt(5)$. Suppose that δ is at distance 2 from a. Then $\triangle^3(\alpha) = \emptyset$, thus $\Gamma = \{\alpha\} \cup \triangle(a) \cup \triangle^2(\alpha)$ and $|\Gamma| = 100$. It is now easy to see that Γ is isomorphic to the Higman-Sims graph (see for example [19] for a formal proof). As $|H| = 100 \cdot |M_{22}| = |HS|$, $H \cong HS$.

So we may assume from now on that δ is not in distance 2 from α. It follows that

2 *$\delta \in \triangle^3(\alpha)$ and H acts transitively on geodesics of length 3.*

By Lemma 3.2 part (h), $H_{\alpha\gamma\delta}$ has orbits of lengths 1, 6 and 15 on $\triangle(\delta)$. Further by (2), $H_{\alpha\delta}$ acts transitively on $\triangle^2(\alpha) \cap \triangle(\delta)$. Since $\triangle(\beta) \cap \triangle(\delta) \subset \triangle^2(\alpha) \cap \triangle(\delta)$, we get $|\triangle^2(\alpha) \cap \triangle(\delta)|$ is 7, 16 or 22.

Suppose $|\triangle^2(\alpha) \cap \triangle(\delta)| = 7$. Then $|H_{\alpha\delta}| = 7 \cdot |H_{\alpha\gamma\delta}|$ and $H_{\alpha\delta} \cong Alt(7)$. Now $H_{\alpha\gamma} \sim 2^4.Alt(6)$ and $H_{\alpha\gamma} \cap H_{\alpha\delta} = H_{\alpha\gamma\delta} \cong Alt(6)$. This contradicts Lemma 3.2, part (d).

Suppose that $|\triangle^2(\alpha) \cap \triangle(\delta)| = 22$. Then $|H_{\alpha\delta}| = 22 \cdot |H_{\alpha\gamma\delta}| = 2^4 \cdot 3^2 \cdot 5 \cdot 11$ and $|H_\alpha/H_{\alpha\delta}| = 56$, a contradiction to Lemma 3.2, part (i).

Thus $|\triangle^2(\alpha) \cap \triangle(\delta)| = 16$ and $|H_{\alpha\delta}| = 16 \cdot |Alt(6)|$. Since $H_{\alpha\delta}$ acts non-trivially on the six points in $\triangle(d) \setminus \triangle^2(\alpha)$, we conclude that $H_{\alpha\delta}$ has a factor group $Alt(6)$ or $Sym(6)$. This implies $H_{\alpha\delta} \sim 2^4 Alt(6)$ and $\triangle(\delta) \setminus \triangle^2(\alpha)$ is a hexad in the $H_\delta-$invariant Steiner system $\triangle(\delta)$. Since $H_{\alpha\gamma\delta}$ has orbits of lengths 1 and 15 on $\triangle^2(\alpha) \cap \triangle(\delta)$, the group $H_{\alpha\beta\gamma\delta}$ ($\cong Alt(5)$) fixes a unique point in $\triangle^2(\alpha) \cap \triangle(\delta)$, namely γ. Note that $H_{\alpha\beta\delta} \cong H_{\alpha\gamma\delta} \cong Alt(6)$ and that $H_{\alpha\beta\delta}$ does not fix γ. So $H_{\alpha\beta\delta}$ fixes no point in $\triangle^2(\alpha) \cap \triangle(\delta)$ and by Lemma 3.2, part (g), $H_{\alpha\beta\delta}$ has orbits of lengths 6 and 10 on $\triangle^2(\alpha) \cap \triangle(\delta)$. Now parts (e) and (f) of Lemma 3.2 imply that $H_{\alpha\beta\gamma\delta}$ has orbits of lengths 1, 5 and 10 on $\triangle^2(\alpha) \cap \triangle(\delta)$ and acts transitively on $\triangle(\delta) \setminus \triangle^2(a)$. Let $\varepsilon \in \triangle(d) \setminus \triangle^2(\alpha)$. Then $H_{\alpha\delta\varepsilon} \sim 2^4 Alt(5)$. Since $H_{\alpha\delta\varepsilon}$ lies in a unique subgroup $2^4.Alt(6)$ of H_α and since the stabilizer in H_α of points at distance 3 from α are $2^4 Alt(6)$'s, we get that δ is the unique point in $\triangle^3(\alpha)$ fixed by $H_{\alpha\delta\varepsilon}$. In particular $\varepsilon \notin \triangle^3(\alpha)$. We have proved

3 $H_{\alpha\delta} \sim 2^4Alt(6),\ |\triangle^3(\alpha)| = 77,\ \varepsilon \in \triangle^4(\alpha),\ H_{\alpha\delta\varepsilon} \sim 2^4Alt(5),\ H_{\alpha\beta\gamma\delta\varepsilon} \cong D_{10}$ *and* H *acts transitively on geodesics of length 4.*

By (1) and part (b) of Lemma 3.2, a subgroup $2^4Alt(5)$ of $H_{\beta\gamma}$ which has orbits of lengths 1, 1 and 20 on $\triangle(\beta)$ has orbits of lengths 1, 5 and 16 on $\triangle(\gamma)$. Note that $\triangle(\delta) \setminus \triangle^2(a)$ is an orbit of length 16 for $H_{\alpha\delta\varepsilon}$ on $\triangle(\delta)$, and so $H_{\alpha\delta\varepsilon}$ has orbits of lengths 1, 1 and 20 on $\triangle(\varepsilon)$. Since $H_{\alpha\varepsilon}$ acts transitively on $\triangle^3(\alpha) \cap \triangle(\varepsilon)$ and since $\triangle(\gamma) \cap \triangle(\varepsilon) \subset \triangle^3(\alpha) \cap \triangle(\varepsilon)$, we get that $|\triangle^3(\alpha) \cap \triangle(\varepsilon)|$ is 21 or 22. Suppose that $|\triangle^3(\alpha) \cap \triangle(\varepsilon)| = 22$, then $|H_{\alpha\varepsilon}| = 22 \cdot |H_{\alpha\delta\varepsilon}| = 22 \cdot |2^4Alt(5)|$ and thus $|H_\alpha / H_{\alpha\varepsilon}| = 21$, a contradiction. Thus $|\triangle^3(\alpha) \cap \triangle(\varepsilon)| = 21$ and $\triangle(\varepsilon) \setminus \triangle^3(\alpha) = \{\eta\}$ for some η. Thus $H_{\alpha\varepsilon} = H_{\alpha\varepsilon\eta} \cong L_3(4)$. Thus $|\triangle^4(\alpha)| = 22$ and ε is the unique point in $\triangle^4(\alpha)$ fixed by $H_{\alpha\varepsilon\eta}$ and so $\eta \notin \triangle^4(\alpha)$. Hence

4 $H_{\alpha\varepsilon} \cong L_3(4),\ |\triangle^4(\alpha)| = 22,\ \eta \in \triangle^5(\alpha),\ H_{\alpha\varepsilon\eta} = H_{\alpha\varepsilon}$ *and* H *is transitive on geodesics of length 5.*

Since $H_{\alpha\eta}$ is transitive on $\triangle^4(\alpha) \cap \triangle(\eta)$ and $\triangle(\delta) \cap \triangle(\eta) \subset \triangle^4(\alpha) \cap \triangle(\eta)$ we conclude that $\triangle^4(\alpha) \cap \triangle(\eta) = \triangle(\eta)$ and $H_{\alpha\eta} = H_\alpha \cong M_{22}$. Thus

5 $H_{\alpha\eta} = H_\alpha,\ \triangle^5(\alpha) = \{\eta\},\ \Gamma = \sum_{0\le i\le 5} \triangle^i(\alpha)$ *and* $|\Gamma| = 200$*.*

Let ϕ be the map that sends a vertex μ in Γ to the unique point at distance 5 from μ. Then ϕ is obviously a bijection and $(\mu^\phi)^g = (\mu^g)^\phi$ for all μ in Γ and g in $\mathrm{Aut}(\Gamma)$. In particular $\{\alpha^\phi, \beta^\phi\}$ is the set of fixed-points of $H_{\alpha\beta}$ on Γ. So β^ϕ is adjacent to α^ϕ and therefore ϕ is a graph automorphism of Γ. Put $\Gamma_0 = \Big\{\{\mu, \mu^\phi\} \mid \mu \in \Gamma\Big\}$ and let $\{\mu, \mu^\phi\}$ be adjacent to $\{\lambda, \lambda^\phi\}$ if μ is adjacent to λ or λ^ϕ. Then $|\Gamma_0| = 100$, M has orbits of lengths 1, 22 and 77 on Γ_0 and H acts transitively on Γ_0. So as above Γ_0 is the Higman-Sims graph and $\mathrm{Aut}(\Gamma_0) \cong \mathrm{Aut}(HS)$. Let N be the kernel of the action of $\mathrm{Aut}(\Gamma)$ on Γ_0. We claim that $N = \{1, \phi\}$. Indeed, let $n \in N$. Then $\alpha^n \in \{\alpha, \alpha^\phi\}$ and replacing n by $n\phi$ if necessary we may assume that $\alpha^n = \alpha$. Since β^ϕ is not adjacent to α, we have $\beta^n = \beta$. So n fixes the neighbors of all its fixed-points and since Γ is connected, we conclude that $n = 1$. Thus $N = \{1, \phi\}$ and $\mathrm{Aut}(\Gamma) \cong C_2 \times \mathrm{Aut}(HS)$. Further, $|H| = 200 \cdot |M| = 2 \cdot |HS|$ and so $H \cong \mathrm{Aut}(HS)$ or $C_2 \times HS$ and the lemma is proved.
QED

Lemma 3.5 *(a)* G *has a subgroup* T *of order 3 with* $N_G(T) \sim SU_6(5) \circ SU_3(5)).Sym(3)$*.*

(b) Put $U = N_G(T)$, $U_1 = [V, T]$ *and* $U_2 = C_V(T)$*. Then* $V = U_1 \oplus U_2$*, moreover,* U_1 *and* U_2 *are irreducible as* U *modules,* $U_1 \cong W_6 \otimes_{GF(25)} W_3$ *and* $GF(25) \otimes_{GF(5)} U_2 \cong \bigwedge^3 W_6$*, where* W_i *is the natural* $i-$*dimensional* $GF(25)-$*module for* $SU_i(5)$*, for i=3,6.*

(c) $N_G(U_1) = U$*.*

Proof: It is clear from the extended Dynkin diagram of type E_7 that $2.E_7(\mathbf{K})$ has a subgroup $H \cong SL_3(\mathbf{K}) \circ SL_6(\mathbf{K})$. Moreover, the central involution of the Weyl group induces a graph automorphism on both of the factors and so an application of Lang's theorem yields a subgroup $(SU_6(5) \circ SU_3(5)).Sym(3)$ in $E_7(5)$. Using the embedding of $\mathbf{K}^{1+56}2.E_7(\mathbf{K})$ in $E_8(\mathbf{K})$, the Steinberg relations and weight theory it is easy to check that the 56−dimensional $\mathbf{K}E_7(\mathbf{K})-$module is as an $H-$module the direct sum of $X_3 \otimes X_6$, $X_3^* \otimes X_6^*$ and $\bigwedge^3 X_6$ where X_i is a natural module for $SL_i(\mathbf{K})$, $i = 3, 6$. As H is a maximal connected closed subgroup of $2.E_7(\mathbf{K})$ and is of index two in its normalizer, all the statements of the lemma are now readily verified.

Lemma 3.6 *There exists an involution* $\bar{f}$ *in* $\overline{U}\backslash\overline{U}'$ *such that* $C_{\overline{U}}(\bar{f}) \sim L_4(5).2^2 \times Sym(5) \times \langle\bar{f}\rangle$. *Moreover, for any such* $\bar{f}$,

$$C_{\overline{G}}(\bar{f}) \sim (2 \times L_8(5)).2$$

and $C_{\overline{U}}(\bar{f})$ *acts transitively on the elementary abelian subgroups of order 16 contained in the normal subgroup* $L_4(5)$ *in* $C_{\overline{U}}(f)$.

Proof: It is easy to verify that $\mathrm{Aut}(U_6(5))$ has three classes of involutions that do not induce a diagonal automorphism on $U_6(5)$. The derived groups of the respective centralizers in $U_6(5)$ are $PSp_6(5)$, $D_3(5)$ and ${}^2D_3(5)$, and the first two of these three classes of involutions lie in the same coset of $\mathrm{Inn}(U_6(5))$.

Since $\Omega_6^-(5)$ ($\cong 2.{}^2D_3(5)$) acts absolutely irreducibly on the exterior cube of its natural module, we conclude that the centralizer of $\Omega_6^-(5)$ on the 20−dimensional module lies in the center of the full linear group acting on the 20−space. It follows that $\overline{U} \setminus \overline{U}'$ contains no involution centralizing a ${}^2D_3(5)$ in $U_6(5)$ and so contains an involution centralizing a $D_3(5)$ in $U_6(5)$.

Let $\bar{f}$ be any such involution. Note that the normalizer of $D_3(5)$ in $U_6(5)$ is $PSO_6{}^-(5)$ extended by an involution which multiplies the quadratic form associated to $PSO_6{}^-(5)$ by a fourth root of unity. So $C_{U_6(5)}(\bar{f}) \sim L_4(5).2^2$ and the two classes of elementary abelian subgroups of order 16 in $C_{U_6(5)}(\bar{f})'$ are fused in $C_{U_6(5)}(\bar{f})$. Moreover, $\mathrm{Aut}(U_3(5))$ has exactly one class of involutions outside $\mathrm{Inn}(U_3(5))$ and the corresponding centralizers are $Sym(5)$'s. Therefore $C_{\overline{U}'}(\bar{f}) \sim L_4(5).2^2 \times Sym(5)$.

It remains to determine $C_{\overline{G}}(\bar{f})$. Note that $\overline{G}$ has three classes of involutions whose centralizers have shapes $2.(L_2(5) \times D_6(5)).2$, $(C_2 \times E_6(5)).2$, and $(C_2 \times L_8(5)).2$. Under the actions of the derived groups of these centralizers V decomposes into direct sums of irreducible modules of dimensions 24 and 32; 1, 1, 27, and 27; and 28 and 28, respectively. On the other hand, from the action on the 20− and 36−spaces, we know that V is the direct sum of irreducible modules of dimensions $18, 18, 10$ and 10 for $C_U(f)'$. It follows that $C_{\overline{G}}(\bar{f}) \sim (C_2 \times L_8(5)).2$, and the lemma is proved. **QED**

From now on let T, U and f be as in Lemma 3.6. Note that f^2 is the central involution of G. Put $N = C_U(f)$, $F = \langle f \rangle$ and $R = N_G(F)$. Then $\overline{R} = C_{\overline{G}}(\bar{f})$ and so by Lemma 3.6 $R \sim C_4 \circ 2 \cdot L_8(5).2$. Let $\widehat{R}'$ be a group with $\widehat{R}' \cong SL_8(5)$ so that $\widehat{R}'$ has R' as a quotient group. For X in R', let $\widehat{X}$ be the inverse image of X in $\widehat{R}'$. As $Z(\bar{N}') = 1$, it is easy to see that the natural 8−dimensional module for $\widehat{R}'$ is, as a module for $\hat{N}'$, the tensor product of a 4−dimensional module for $SL_4(5)$ and a 2−dimensional module for $SL_2(5)$. Since elements in $GL_4(5) \otimes GL_2(5) \leq GL_8(5)$ have determinant plus or minus one, $\overline{R}'$ has two classes of subgroups isomorphic to $L_4(5) \times L_2(5)$. Since $R = NR'$, R fixes these classes. Moreover, as the element inverting f can be chosen to invert a Cartan subgroup of R', R does not induce an outer diagonal automorphism on R'.

The character tables of $L_3(4)$ and its covers (see [3]) show that there is a group of shape $4.L_3(4)$ which has a faithful irreducible character χ of degree 8. Note that R contains a subgroup L of shape $2.L_3(4)$. We plan to extend L to a subgroup M_{22} and then to a subgroup HS of G. Let S be a Sylow 2−subgroup of L, let $B = N_L(S)$ and let A_1 and A_2 be the two elementary abelian groups of order 2^5 in S. Note that $\widehat{L}$ is perfect and that $\widehat{A}_i$ is the central product of a cyclic group of order 4 with an extra-special group of order 2^5. Moreover a natural 8−dimensional module for $\hat{R}'$ is as an $\widehat{A}_i$−module the direct sum of two isomorphic irreducible modules of dimension 4. Thus $C_{\widehat{R}'}(\widehat{A}_i) \cong GL_2(5)$ and we can choose L, S and A_1, so that $\overline{A}_1$ is contained in the normal $L_4(5)$ in $\overline{N}$. It follows that $N_{\overline{R}'}(\overline{A}_1) \le N$ and has shape $2^4 Sym(6) \times Sym(5)$. Let $\overline{B}_2$ be the projection of A_2 onto the $L_4(5)$ in N. Let $\overline{B}_2$ be the projection of A_2 onto the $L_4(5)$ in N. Put $L_i = N_L(A_i)$. Our goal is to show that both L_i's can be extended to a subgroup $2^5 Sym(6)$ in U. Inside $2^5 Sym(6)$ we will choose appropriate subgroups $2^5 Sym(5)$ and $2^5 Alt(6)$ which will allow us to apply Lemma 3.3 and Lemma 3.4 to find M_{22} and HS in $\overline{G}$.

Lemma 3.7 *In R', the subgroup A_2 is conjugate to B_2 but not to A_1.*

Proof: Put $C_i = O^2(C_{\widehat{R}'}(\widehat{A}_i))$. Then $C_i \cong SL_2(5)$. Suppose that $[S, C_1] = 1$. Then $C_1 \le C_{\widehat{R}'}(\widehat{A}_2)$ and $C_1 = C_2$. Since $L = \langle L_1, L_2 \rangle$, the group L normalizes C_1, a contradiction. Hence $[S, C_1] \neq 1$ and similarly $[S, C_2] \neq 1$. As $B/A_i \cong Alt(4)$, we have $B = O^2(B)A_i$. Since $Out(C_i)$ is a 2−group and A_i centralizes C_i we conclude that S induces inner automorphisms on C_i. Hence S normalizes a unique subgroup Q_8 in C_i. Denote this Q_8 by Q_i and note that S induces inner automorphisms on Q_1. In particular $[\widehat{A}_2, Q_1] \le Z(Q_1) \cdot Z(\widehat{R}')$. Thus Q_1 induces inner automorphisms on $\widehat{A}_2$. Put $R_i = Q_1 \widehat{A}_i$ (where we really mean Q_1 and not Q_i). Then R_1 and R_2 are both the central product of an extra-special group of order 2^7 with a cyclic group of order 4. Put $X = C_{R_2}(\widehat{A}_2)$. It follows that $R_2 = X\widehat{A}_2$, that S normalizes X and that X contains a Q_8 invariant under S. Thus $Q_2 \le X$, $R_2 = Q_2\widehat{A}_2 = Q_2 B_2$ and $\hat{B}_2 = C_{R_2}(Q_1)$.

Note that $N_{\widehat{R}'}(R_1)/R_1 \cong Sp_6(2)$. Note $\bar{A}_1$ and $\bar{B}_2$ are both nondegenerate 4−spaces in the 6−dimensional symplectic space $R_1/Z(R_1)$ (where the symplectic form is given by the commutator map). Hence by Witt's theorem, A_1 and B_2 are conjugate under $N_{\widehat{R}'}(R_1)$ proving the first statement of the lemma.

Since $C_{\widehat{R}'}(\widehat{A}_i)$ acts transitively on the Q_8's in $C_{\widehat{R}'}(\widehat{A}_i)$ and since $N_{\widehat{R}'}(A_1) \cap N_{\widehat{R}'}(Q_1)$ induces the full automorphism group of Q_1 on Q_1, we conclude that A_1 and B_2 are conjugate in R' if and only if they are conjugate under $Y = C_{\widehat{R}'}(Q_1)$. Note that Y is isomorphic to the subgroup of index two in $GL_4(5)$. Since B normalizes Y and $[A_1, B] = A_1$, A_1 and hence also B_2 are contained in Y'. Put $S_0 = A_1 B_2$. Since A_1 and B_2 normalize each other we conclude that S_0 is a Sylow 2−subgroup of $Y/Z(Y)$ and that A_1 and B_2 are the two maximal elementary abelian subgroups of S_0. It follows that A_1 and B_2 are not conjugate in Y'. Moreover, $\mathrm{Out}(Y') \cong D_8$, $\mathrm{Out}(Y')$ acts on $\{A_1^{Y'}, B_2^{Y'}\}$ and $\mathrm{Out}(Y')'$ fixes $A_1^{Y'}$ and $B_2^{Y'}$. As the group of outer automorphisms of Y' induced by Y is $\mathrm{Out}(Y')'$, A_1 and B_2 are not conjugate in Y. So A_1 and B_2 are not conjugate in R' and lemma is established. **QED**

Lemma 3.8 *$N_{\overline{R}'}(\overline{A}_1)$ acts transitively on $L_3(4)$'s in $\overline{R}'$ containing $\overline{A}_1$ and on subgroups $2^4 Alt(5)$ in $N_{\overline{R}'}(\overline{A}_1)$ which can be extended to an $L_3(4)$ in $\overline{R}'$.*

Proof: Recall that $N_{\overline{R}'}(\overline{A}_1) \sim 2^4 Sym(6) \times Sym(5)$ and, as $[S, C_i] \neq 1$ in the notation of the previous lemma, any $2^4 Alt(5)$ in $N_{\overline{R}'}(\overline{A}_1)$, which can be extended to a $L_3(4)$ in $\overline{R}'$, projects non-trivially on the $Sym(5)$. Hence $N_{\overline{R}'}(\overline{A}_1)$ acts transitively on such subgroups. Note that $\overline{L}_1$ is such a subgroup and that $L_1^* \stackrel{def}{=} N_{\overline{R}'}(\overline{L}_1) \sim 2^4 Sym(5)$. Put $\overline{W} = N_{\overline{R}'}(\overline{L}_1) \cap N_{\overline{R}'}(\overline{A}_2)$. As $\bar{A}_1$ and $\bar{A}_2$ are the only maximal elementary abelian subgroups of $\bar{S} = \bar{A}_1 \bar{A}_2$, we have

$N_{L_1^*}(A_2) = N_{L_1^*}(S)$. Since S/A_1 is the Sylow $2-$subgroup of L_1/A_1, we have $\overline{W} \sim 2^4 Sym(4)$. We claim that W is contained in a $2^4 Alt(6) \times Sym(5)$ subgroup of $N_{\overline{R}'}(\overline{A}_1)$. For this let i be an involution in $\overline{W} \setminus \overline{L}_1$. Then it is enough to show that the inverse image of $[\widehat{A}_2, i]$ is not elementary abelian. Let C_2 be the projection of A_2 onto the normal $Sym(5)$ in $\overline{N}$. Then $[\widehat{B}_2, i]$ is elementary abelian, while $[\widehat{C}_2, i]$ is not. So also $[\widehat{A}_2, i]$ is not elementary abelian, and the claim is proved.

It is now easy to see that $\overline{W} \cap \overline{L}_1$ is contained in exactly two subgroups $2^4 Alt(5)$ of $N_{\overline{R}'}(\overline{A}_2)$ and these two subgroups are interchanged by W. Since any $L_3(4)$ in $\overline{R}'$ containing $\overline{L}_1$ is generated by $\overline{L}_1$ and a $2^4 Alt(5)$ in $N_{\overline{R}'}(\overline{A}_2)$ containing $\overline{W} \cap \overline{L}_1$ the lemma is proved. **QED**

Lemma 3.9 *Put $L_0 = N_R(L)$. Then $|L_0/LF| = 2$ and A_1 and A_2 are conjugate in L_0.*

Proof: Since $\mathrm{Out}(R'/Z(R')) \cong D_8$, since R' has four orbits on $A_1^{\mathrm{Aut}(R')}$ and since $PGL_8(5)$ acts transitively on those four orbits, the normalizer of any such orbit in $\mathrm{Out}(R')$ is a group of order 2 that is not contained in the center of $\mathrm{Out}(R')$. It follows that R fixes two of these orbits and interchanges the remaining two. Moreover, by Lemma 3.8, $L_8(5)$ has four orbits on $(A_1, L)^{\mathrm{Aut}(R')}$ and, by Lemma 3.7, each orbit of subgroups $L_3(4)$ in $\overline{R}'$ leads to two orbits of R' on $(A_1, L)^{\mathrm{Aut}(R')}$. It follows that $\overline{R}'$ has exactly two classes of subgroups $L_3(4)$. Suppose R interchanges those two classes. Then R could not normalize any of the orbits of R' on $A_1^{\mathrm{Aut}(R')}$, a contradiction. So $|L_0/LF| = 2$. Suppose that A_1 and A_2 are not conjugate in L_0. Then they are also not conjugate in R. By Lemma 3.7, we conclude that A_1 and B_2 are not conjugate in R, and this contradicts Lemma 3.6. So A_1 and A_2 must be conjugate in L_0. **QED**

Lemma 3.10 *Put $X = N_G(A_1)$. Then $X \leq U$ and $X \sim 6.(2^4.3^4.Sym(6) \times U_3(5)).Sym(3)$.*

Proof: Let $X_0 = N_U(A_1)$. By the action of A_1 on the natural $6-$dimensional $GF(25)-$module W for $SU_6(5)$ we see that, modulo the normal $SU_3(5)$ in U, X_0 is a full monomial subgroup of $U_6(5).Sym(3)$. It follows that $X_0 \sim 6.(2^4.3^4.Sym(6) \times U_3(5)).Sym(3)$. Moreover, X_0 has two orbits Σ_1 and Σ_2 on the hyperplanes in A_1 which do not contain $Z(G)$. Choose notation so that $|\Sigma_1| = 6$ and $|\Sigma_2| = 10$. Recall that $V \cong U_1 \oplus U_2$ where U_1 and U_2 are defined and described in Lemma 3.5.

Let H_1, H_2, H_3 be three different elements of Σ_1. Then its is easy to check that $H_1 \cap H_2 \not\leq H_3$. As $W = \bigoplus_{H \in \Sigma_1}(C_W(H))$, this implies that H_1 acts fixed–point freely on U_2. Since U_1 is a direct sum of copies of W as an A_1-module, we get $\bigoplus_{H \in \Sigma_1} C_V(H) = U_1$.

Also $C_V(H)$ is either $6-$dimensional, if $H \in \Sigma_1$, or $20/10 = 2-$dimensional, if $H \in \Sigma_2$. Thus $N_G(A_1)$ normalizes Σ_1 and so also U_1. Since U is maximal in G, we conclude that $X \leq U$. Hence $X = X_0$. **QED**

Lemma 3.11 *There exists $t \in N_R(L)$ and $D \leq U$ such that $t^2 \in L$, $A_1^t = A_2$, $L_1 \leq D$, $D \sim 2^5 Sym(6)$, t normalizes $N_D(B)$ and t does not normalize $N_{D'}(B)$.*

Proof: Let $Y = X/A_1$. Then $Y \sim 3.(3^4 Sym(6) \times U_3(5))Sym(3)$. Let K be the image of L_1 in Y. Then $K \cong Alt(5)$ and we are looking for subgroups $Sym(6)$ of Y containing K. Let

Y_1 be the normal subgroup $3.3^4 Sym(6)$ of Y, let Y_2 be the normal subgroup $SU_3(5)$ of Y and let K_i be the projection of K onto Y_i. Then Y acts transitively on subgroups $3 \cdot Alt(6)$ (the triple cover of $Alt(6)$) and transitively on subgroups $Alt(5)$ in Y_2. Moreover, the normalizer of an $Alt(5)$ in $U_3(5).Sym(3)$ is a $C_2 \times Sym(5)$ and the normalizer of an $Alt(6)$ is $Aut(Alt(6))$. It follows that K_2 can be embedded into exactly two subgroups $3 \cdot Alt(6)$ of Y_2 and f interchanges these two $3 \cdot Alt(6)$'s. Furthermore Y has one orbit on subgroups $3 \cdot Alt(6)$ in Y_1 and one orbit on subgroups $Alt(5)$ in Y_1 for which $\overline{A}_1$ is a natural $SL_2(4)-$module. (Note here that $3 \cdot Alt(6)$ exists in Y_1, since $3^4 Alt(6)$ has 9 classes of $Alt(6)$'s while $3^{1+4} Alt(6)$ has only 3 classes.) Now $Y/Y_2 \sim 3^{4+1}(C_2 \times Sym(6))$. The normalizer of the image of K in Y/Y_2 is a $C_2 \times Sym(5)$ and the normalizer of the image of a $3 \cdot Alt(6)$ of Y_1 in Y/Y_2 is a $Sym(6)$. It follows that K_1 can be embedded into exactly two subgroups $3 \cdot Alt(6)$ of Y_1 and f interchanges these two $3 \cdot Alt(6)$'s. Let D_i be any of the two subgroups $3 \cdot Alt(6)$ in Y_i with $K_i \leq D_i$. Then there exists precisely one subgroup A in $D_1 D_2$ such that $K \leq A$, $A = A'$ and $A/Z(A) \cong Alt(6)$. Similarly there exists precisely one subgroup $\widehat{A}$ in $D_1^f D_2$ such that $K \leq \widehat{A}$, $\widehat{A} = \widehat{A}'$ and $\widehat{A}/Z(\widehat{A}) \cong Alt(6)$. We claim that exactly one of A and $\widehat{A}$ is an $Alt(6)$, while the other is the triple cover. Indeed, let x_i, y_i be elements of order 3 in D_i such that $\langle x_1 x_2, y_1 y_2 \rangle Z(Y)$ is a Sylow $3-$subgroup of $AZ(Y)$. Then $\langle {x_1}^f {x_2}^f, y_1 y_2 \rangle Z(Y)$ is a Sylow $3-$subgroup of $\widehat{A}Z(Y)$. Note that

$$[x_1 x_2, y_1 y_2] = [x_1, y_1]\,[x_2, y_2] \quad \text{and} \quad [{x_1}^f {x_2}^f, y_1 y_2] = [x_1, y_1]^f\,[x_2, y_2]. \qquad (*)$$

Since $[x_1, y_1]$ and $[x_2, y_2]$ are both contained in $Z(Y)$ and unequal to 1 and since f inverts $Z(Y)$, we see that exactly one of the two expressions in $(*)$ is equal to 1. This proves our claim.

Choose notation so that $A \cong Alt(6)$. From what we proved so far it follows that A and A^f are the only two $Alt(6)$ subgroups in Y which contain K. It is easy to see that $N_Y(K)/Z(Y) \cong C_2 \times Sym(5)$. Furthermore, there exists a subgroup of index 2 in $N_Y(K)$ which normalizes A. Since f does not normalize A, this subgroup is, modulo the center of Y, isomorphic to $Sym(5)$. Put $E = A(N_Y(K) \cup N_Y(A))$ and let $\widehat{D}$ be the inverse image of E in X. Then $E/Z(Y) \cong Sym(6)$ and $\widehat{D}/T \cong 2^5 Sym(6)$. (Recall that T is the cyclic group of order three with $U = N_G(T)$.)

By Lemma 3.9, there exists $t \in N_R(L)$ with ${A_1}^t = A_2$. Choose t so that t normalizes B. Then $t^2 \in BF$ and since t inverts F, we have $t^2 \in B$. In particular $t^2 \in L$. Note that A_1 and A_2 are the only elementary abelian groups of order 2^5 in B. Put $J = N_G(B) \cap N_G(A_1)$. As $t \in N_G(B) \setminus J$, we conclude that J is of index two in $N_G(B)$. For $Q \subset N_G(B)$ let Q^* be the image of Q in $N_G(B)/B$. Since $J \leq X$, it is easily verified that J^* is an elementary abelian group of order 9 extended by an elementary abelian group of order 8. Pick a in $N_{\widehat{D}'}(B) \setminus BT$ and b in $N_{\widehat{D}}(B)$ with $[a, B] \leq A_1$ and such that a^* and b^* are involutions. Since $U_3(5)$ contains no $Sym(6)$, b induces an outer automorphism on Y_2. The same is true for f and so b and f both invert $O_3(J^*)$. Note that a centralizes T and inverts $O_3(J^*)/T$. As A_1 is in the $SU_6(5)$ subgroup of U but A_2 is not, the element t is not in U. Since $T \leq O_3(J)$, we conclude $O_3(J)^* = T^* T^{*t} = O_3(J^*)$. It follows that $\langle a^*, t^* \rangle$ acts as a D_8 on $O_3(J^*)$. Since $\langle a^*, t^* \rangle$ is a dihedral group, we conclude that $\langle a^*, t^* \rangle \cong D_8$. Let $x^* = [a^*, t^*]$.

We claim that $x^* T^* = b^* T^*$. Indeed, x^* inverts $O_3(J^*)$ and $|C_{N_G(B)^*}(O_3(J)^*)/O_3(J^*)| = 2$. Hence x^* lies in the same coset of $O_3(J^*)$ as f^* or as b^*. Since $N_{\bar{R}'}(A_1) \sim 2^4.Sym(6) \times Sym(5)$, $N_{R'}(B)^* \cong C_2 \times C_2$. Furthermore f and t are in $N_R(B)$ and so $N_R(B)^*$ is a Sylow $2-$subgroup of $N_G(B)^*$. Since $f \notin R'$ we conclude that $f^* \notin N_G(B)^{*\prime} O_3(J^*) = \langle x^* \rangle O_3(J^*)$. So $x^* O_3(J^*) = b^* O_3(J^*)$. Moreover, a^* centralizes b^* and x^* and hence $b^* T^* = C_{x^* O_3(J^*)}(a^*) = x^* T^*$.

In particular, we can choose b so that $b^* = x^* = [a^*, b^*]$. Put $D = \widehat{D}' \langle b \rangle$. Then $\overline{D} \sim 2^4 Sym(6)$. In addition, $N_D(B) = B \langle a, b \rangle$ and so t normalizes $N_D(B)$. On the other hand, $N_{D'}(B) = B\langle a \rangle$ and so $N_{D'}(B)^t = B\langle ab \rangle \neq N_{D'}(B)$. **QED**

Proof of Theorem 3.1:

We are now able to construct HS and M_{22} in $\bar{G}$. For this choose L, t and D as in Lemma 3.11 and put $M_1 = N_D(L_1)$. By Lemma 3.10, $N_{D'^t}(B)$ is contained in D, but not in D'. Since $N_{D'^t}(B)/O_2(D) \cong Sym(4)$, we conclude that $N_{D'^t}(B)$ is contained in M_1. Note that $L_1^t = L_2$ and so $L_2 \leq D'^t$. Put $M_2 = D'^t$. Then $\overline{M}_1 \sim 2^4 Sym(5)$ and $\overline{M}_2 \sim 2^4 Alt(6)$. Moreover,

$$M_1 \cap L = L_1, \; M_2 \cap L = L_2, \quad \text{and} \quad M_1 \cap M_2 = N_{D'^t}(B).$$

Put $M = \langle L, M_1, M_2 \rangle$. Then by Lemma 3.3, $\overline{M} \cong M_{22}$. Now $D \cap M = M_1$, $D^t \cap M = M_2$ and $D \cap D^t = N_D(B)$. Let $H = \langle D, L, t \rangle$. Then by Lemma 3.4, $\bar{H} \cong HS, C_2 \times HS$ or $Aut(HS)$. Thus in any case $\bar{H}' \cong HS$ and Theorem 3.1 is proved. The dedicated reader might check that actually $\bar{H}$ itself is already the Higman-Sims group.

4 A Computational proof that HS is a Subgroup of $E_7(5)$.

In this section we give a computer dependent proof that $HS < E_7(5)$. Our strategy is to use a machine calculation to prove that HS acts (absolutely) irreducibly on a $133-$dimensional, $5-$modular, Lie algebra. We then apply the extensive theory of modular Lie algebras to deduce that the Lie algebra is simple, that it is a classical modular Lie algebra of type E_7 over $GF(5)$, and thence $HS < E_7(5)$.

Computation 4.1 *We construct an explicit* $133-$*dimensional (absolutely) irreducible matrix representation of* HS *over* $GF(5)$.

Method: We remark that although it seems natural to construct the $133-$dimensional representation as a constituent of a tensor product of smaller representations of HS, all useful tensor products are too large for our implementation of the meataxe. For example, the $133-$dimensional, $5-$modular representations of HS are constituents in the symmetric cube of the $21-$dimensional representation and in the tensor product of a $21-$dimensional representation with a $55-$dimensional representation, however these representations have degrees 1771 and 1155. In order to avoid such large computations, we shall locate a $133-$dimensional representation as a constituent of the symmetric square of a $28-$dimensional representation of the double cover of HS: this latter representation can be found in a previously known representation of the Harada–Norton group.

We start with the $133-$dimensional matrix representation of HN over $GF(5)$ that is constructed in [15]. We locate matrices, x_1 and y_1, that represent HN elements of classes $40A$ and $12C$ (since these classes have small centralizers, we locate such elements by a random search). Let $x = {x_1}^{20}$. Every dihedral group generated by the $2A-$element x and a conjugate of the

$2B-$element ${y_1}^6$ has a central involution. By collecting such involutio ns together with the matrix x_1, it is an easy matter to generate enough matrices to represent the centralizer of x in HN (we just keep adding involutions until we observe a group that contains an element of order 11). The group $C_{HN}(x)$ has structure $2.HS.2$.

We use the meataxe [10] to decompose the restriction of our $HN-$module into its irreducible constituents of degrees 1, 21, 28, 28, and 55, under the group $2.HS = (C_{HN}(x))'$. Another application of the meataxe to the symmetric square of either of the irreducible $28-$dimensional $2.HS-$modules yields an (absolutely) irreducible matrix representation of HS of degree 133. **QED**

Let E denote the $133-$dimensional $HS-$module afforded by the matrix representation of Computation 4.1. In our calculations with E we shall use a fixed basis $e_1, e_2, \ldots, e_{133}$ of E: moreover for technical reasons, we use a basis on which a particular subgroup

$$P \sim 2^4.2^3 < 2^4.(2 \times Sym(4)) < 2^4.Sym(6) < HS$$

acts monomially. We let ${e_1}^*, {e_2}^*, \ldots, {e_{133}}^*$ denote the dual basis of the dual module E^* (this module is isomorphic to E, but for computational purposes it is convenient to distinguish E and E^*).

Computation 4.2 *We compute an $HS-$invariant product, $*$: $\Lambda^2 E \to E$.*

Method: The output from this computation is a list of components, $a_{i,j,k}$ of an invariant rank three tensor such that the map $e_i \wedge e_j \mapsto \sum_k a_{i,j,k}\, e_k$ extends to an $HS-$invariant multilinear map. It is convenient to use duality to observe that the map ${e_k}^* \mapsto \sum_{i,j} a_{i,j,k}\, {e_i}^* \wedge {e_j}^*$ extends to an $HS-$invariant multilinear map. Moreover, since the group HS acts irreducibly on E^*, it is easy to use the group action to compute these tensor components once we know the image of any single vector of E^* under an $HS-$invariant map: $E^* \to \Lambda^2 E^*$.

The module $\Lambda^2 E^*$ has degree 8778 and is too large to decompose directly with our implementation of the meataxe: in order to locate all copies of E^* in $\Lambda^2 E^*$ we use the condensation techniques described in [14]. Let π denote the idempotent $(\sum_{p \in P} p)/|P|$ of the group algebra $GF(5)HS$. The condensation programs of [14] compute matrix representations of the Hecke algebra $\pi GF(5)HS\pi$ on the condensed modules $E^*\pi$ and $\Lambda^2 E^*\pi$ (which have degrees 1 and 50). (It is in the computation of these matrix representations of the Hecke algebra that our programs require the group P to act monomially on E and E^*.) A standard meataxe calculation locates the single copy of $E^*\pi$ in the $\pi GF(5)HS\pi-$module $\Lambda^2 E^*\pi$. Thus there is a single embedding of E^* in the $HS-$module $\Lambda^2 E^*$. Moreover, the embedding of Hecke algebra modules gives the image of the $1-$dimensional space of fixed points of P on E^* under the $HS-$invariant map: $E^* \to \Lambda^2 E^*$. As we remarked above, the action of HS now determines the components of the $HS-$invariant tensor $a_{i,j,k}$. **QED**

For each $e \in E$ we write $*e$ for the matrix that represents the action of right multiplication by e on our $HS-$invariant algebra. After completing Computation 4.2, we ran a simple precautionary program to verify $HS-$invariance of our tensor. For each basis vector e_i and for each of generator, h, of HS, we checked that the matrices $(*e_i)^h$ and $*({e_i}^h)$ are identical.

Computation 4.3 *The $HS-$invariant product $*$: $\Lambda^2 E \to E$ of Computation 2 is a Lie product on E. Moreover the Killing form on $(E, *)$ is non-singular.*

Method: A straightforward computation shows that for each basis vector e_i we have

$$(*e_i)(*e_1) - (*e_1)(*e_i) - (*((e_i)(*e_1))) = 0.$$

Hence the Jacobi identity holds for any triple of basis vectors of the form e_1, e_i, e_j. Therefore right multiplication by e_1 is a derivation of $(E, *)$; and, by applying the action of HS, we deduce that $*(e_1^h)$ is a derivation of $(E, *)$ for any choice of $h \in HS$. Since HS is irreducible on E, we deduce that $*$ is a Lie product on E.

A random search quickly produces $e \in E$ with $\mathrm{Tr}((*e)(*e)) \neq 0$. It follows that the Killing form on E is not identically zero: irreducibility of E as a $HS-$module now shows that the Killing form is non-singular. **QED**

We complete the proof that $HS < E_7(5)$ by applying standard results to show that the automorphism group of $(E, *)$ can only be $\mathrm{Aut}(E_7(5))$. The following lemma from [13] shows that $(E, *)$ is a simple Lie algebra.

Lemma 4.4 *Suppose that $(X, *)$ is a finite dimensional (non-associative) algebra and that $A \leq Aut(X, *)$ acts irreducibly on X. Then one of the following holds:*
(a) The algebra $(X, *)$ is simple.
(b) The $A-$module X is induced from a module of a proper subgroup of A.
(c) The product $*$ is identically zero.

Proof: Let I be a minimal non-zero ideal of $(X, *)$, and let $\mathcal{I} = \{I^a | a \in A\}$. We say that a subset of $\mathcal{I}$ is independent if it consists of independent (vector) subspaces of X. Let $\mathcal{J} = \{I_1, I_2, \dots I_l\}$ be a maximal independent subset of $\mathcal{I}$. Let $Y = \oplus_{k=1}^{l} I_k$, then Y is an ideal of $(X, *)$.

Let I^a be any $A-$image of I. Maximality of $\mathcal{J}$ shows that $I^a \cap Y$ is a non-zero ideal; hence, since I^a is a minimal ideal, we have $I^a \subset Y$. Thus, Y contains a non-zero $A-$submodule of X, and, since X is irreducible, we have $X = Y$.

We now suppose that neither (a) nor (b) holds: thus $I \neq X$ and hence $\mathrm{Stab}_A I$ is a proper subgroup of A. Moreover, since X is not induced, there is an $a \in A$ with $I^a \notin \mathcal{J}$. Then, $I^a * I_k \subset I^a \cap I_k = \{0\}$, for each $I_k \in \mathcal{J}$. Thus $I^a * X = I^a * Y = \sum I^a * I_k = \{0\}$. The $A-$invariance of $*$ now gives $I_k * X = \{0\}$ for each $I_k \in \mathcal{J}$, and thus $X * X = \sum I_k * X = \{0\}$.
QED

Let $(\overline{E}, *)$ be the Lie algebra obtained from $(E, *)$ by extending the scalars to the algebraic closure of $GF(5)$. Since E is absolutely irreducible as a $HS-$module and it is not induced (since HS has no subgroup of index 133), Lemma 4.4 shows that the Lie algebra $(\overline{E}, *)$ is simple. Moreover, by Computation 4.3, the simple Lie algebra $(\overline{E}, *)$ has a non-singular Killing form.

Over an algebraically closed field, F say, of characteristic $p > 3$, the modular Lie algebras with a non-singular trace form are completely classified by a theorem of Block and Zassenhaus [1] (this result is also given in [16], page 49). Block and Zassenhaus show that such a modular

Lie algebra is a direct sum of abelian Lie algebras, total matrix algebras $M_n(F)$ where $p|n$, and classical simple Lie algebras of types $A_1, \ldots, E_8$. In particular, our algebra $(\overline{E}, *)$ must be the classical simple Lie algebra of type E_7 (since no other simple algebra in the list provided by [1] has dimension 133). Therefore, the $HS-$invariant algebra $(E, *)$ is a $GF(5)-$form of E_7. By Theorem IV.6.1 of [16] there is just one $GF(5)-$form of E_7 : thus $(E, *)$ is the classical simple Lie algebra of type E_7 over $GF(5)$. It now follows from [18] that $Aut(E, *)$ has structure $E_7(5).2$ and we obtain $HS < E_7(5)$.

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