MTH991 Fall 97
Amalgams, N-Groups and Groups of Characteristic p-Type

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Contents

Chapter 1

Introduction

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These lecture notes are devoted to the study of finite groups via their p-local subgroups. Let p be a prime and G a finite group then a *p-local subgroup* of G is a subgroup of the form $N_G(P)$ where P is a non-trivial p-subgroup of G. Finite groups can often be characterized in terms of their p-local subgroups.

In general given the task to identify a finite group one usually tries to find some object the group is acting on in a reasonable nice way. These object are usually represented by some proper subgroups of the group and looking at p-local subgroups is often the only way or at least the most convenient way to locate proper subgroups of a group.

Before we start to investigate abstractly the p-local structure of finite groups we will have a look at the p-locals of one particular class of groups, the general linear groups. Hopefully this example will show that the p-local structure of a group carries a good amount of information about the group.

Chapter 2

The p-local structure of $GL_K(V)$

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Let K be a finite field in characteristic p and V a finite dimensional vector space over K. Let $n = \dim_K V$. The goal of this chapter is to recover the projective space associate to V in terms of the maximal p-local subgroups of $G = GL_K(V)$. Let P be a non-trivial p-subgroup of G. Put $W = C_V(P)$. The number of vectors in W is $|K|^{\dim W}$ which is a power of p and so is divisible by p. On th otherhand, as P is a p-group, the length of every non-trivial orbit of P on W is divisible by p and so the number of fixed point of P on W equals $|W|$ modulo p. Thus P has to centralize a vector besides the zero vector and so W is a proper subspace of V. Note that $N_G(P) \leq N_G(W)$. Let $Q_W = C_G(W) \cap C_G(V/W)$. Then Q is an elementary abelian subgroup normal subgroup of $N_G(W)$. Since $C_V(Q_W) = W$, $N_G(Q_W) \leq N_G(W)$ and so

$$N_G(P) \leq N_G(Q_W) = N_G(W)$$

We conclude that the maximal p-locals of G are exactly the normalizers of proper subspaces of V and so there exists a one to one correpondence between the maximal p-locals of G and the proper subspaces of V. Recall (or just believe for now) that G is basically the automorphism group of the *projective space* $\mathcal{P}_V$ associated to V. $\mathcal{P}_V$ consists of all the proper subspaces of V together with the incidence relation given by inclusion.

Definition 2.1 d-flag

1. *A* flag $\mathcal{F}$ *in* $\mathcal{P}_V$ *is a chain* $V_1 < V_2 < \ldots < V_r$ *of proper subspaces of* V*.*

2. *The* type *of the flag* $V_1 < V_2 < \ldots < V_r$ *is the* r*-tuple* $(\dim_K V_1, \ldots, \dim_k V_r)$

Lemma 2.2 *$GL_K(V)$ acts transitively on the flags of given type in $\mathcal{P}_V$.* glft

Proof: Let $V_1 < \ldots < V_r$ and $W_1 < \ldots < W_r$ be flags with the same type $(t_1, \ldots, t_r)$. Let $v_1, \ldots, v_n$ be a basis for V so that $v_1, \ldots, v_{t_i}$ is a basis for V_i (start with a basis for V_1 and then extend to a basis for V_2 and so on). Similarly let $w_1, \ldots, w_n$ be a basis for V so that $w_1, \ldots, w_{t_i}$ is a basis for W_i. Define $g \in G$ by $g : v_i \mapsto w_i$. Then $V_i^g = W_i$ and the lemma is established. □

Let us try to recover this incidence relation in terms of the maximal p-locals. For this let X and Y be proper subspaces of V of dimension x and y respectively. The next couple of lemmas are used to show that X and Y are incident if and only if $N_G(X) \cap N_G(Y)$ contains a Sylow p-subgroup of G.

qwis **Lemma 2.3** *Let W be a proper subspace of V.*

(a) *Let U be a Q_W-submodule of V. Then $U \leq W$ or $W \leq U$.*

(b) *W is the unique proper $N_G(W)$-submodule in V.*

(c) $Q_W = O_p(N_G(W))$.

Proof: (a) Let U be any Q_W-submodule of V and assume that $U \not\leq W$. Pick $u \in U \setminus W$. Let $v_1, \ldots, v_n$ be a basis for V so that $v_1, \ldots, v_k$ is a basis for W and $v_n = u$. Let $j \leq k$ and t be the linear map defined by $v_n \mapsto v_n + v_j$ and $v_i \mapsto v_i$ for all $i \neq n$. Then $t \in Q_W$. Also $u^t = u + v_j$. Since Q_W normalizes U we conclude $v_j = u^t - u \in U$. Hence $W \leq U$.

(b) This follows from (a) as $N_G(W)$ acts irreducibly on W and V/W.

(c) Let $Q = O_p(N_G(W))$. Then by (b), $C_V(Q) = W = [V, Q]$ and so $Q \leq Q_W$. □

sylgl **Lemma 2.4** *Let S be a Sylow p-subgroup of G and $0 = V_0 < V_1 < \ldots < V_r = V$ be a composition series for S on V. Then:*

(a) *$(V_1, V_2, \ldots, V_{r-1})$ is a maximal flag, $\dim V_i = i$ and $r = n = \dim V$.*

(b) *$S = \{g \in GL_K(V) \mid [V_i, g] \leq V_{i-1}$ for all $1 \leq i < r\}$.*

(c) *For $0 \leq i \leq n$, V_i is the unique i-dimensional S-submodule in V.*

(d) *Every flag in $\mathcal{P}_V$ is normalized by a Sylow p-subgroup of G.*

Proof: (a) Since S acts irredubibly on V_i/V_{i-1} and $C_{V_i/V_{i-1}}(S) \neq 0$, S centralizes V_i/V_{i-1}. Therefore V_i/V_{i-1} is 1-dimensional. Thus (a) holds.

(b) Let S^* be the group on the right hand side of the equation. Then $S \leq S^*$ and S^* consists of all lower triangular matrices with respect to the basis $v_1, \ldots, v_n$. Thus $|S^*| = |K|^{\frac{n(n-1)}{2}}$ and so S^* is a p-group. As S is a Sylow p-subgroup, $S = S^*$.

(c) Let U be any i-dimensional S-submodule in V. By (b) $Q_{V_i} \leq S$ and so by 2.3a, $U \leq V_i$ or $V_i \leq U$. In any case $U = V_i$ as U and V_i have the same dimension.

(d) holds since by 2.2 every flag is conjugate to a subflag of $V_1 < \ldots < V_{r-1}$. □

XYin **Lemma 2.5** *Let X and Y be proper subspaces of V. Then X and Y are incident if and only if $N_G(X) \cap N_G(Y)$ contains a Sylow p-subgroup of G.*

Proof: Suppose first that $X \leq Y$. Then by 2.4.d $N_G(X) \cap N_G(Y)$ contains a Sylow p-subgroup of G.

Conversely suppose that $N_G(X) \cap N_G(Y)$ contains a Sylow p-subgroup S of G. Then by 2.4.c, $X = V_i$ and $Y = V_j$ for some i and j. So X and Y are incident.

Definition 2.6 dig

1. *An* incidence geometry *is a tuple* $(\mathcal{G}, R, I, t)$ *so that*

   (a) $\mathcal{G}$ *and* I *are sets.*

   (b) R *is a symmetric and reflexive relation on* $\mathcal{G}$*, called the incidence relation.*

   (c) t *is a function from* $\mathcal{G}$ *to* I*, called the type function.*

   (d) *If* a *and* b *are incident and have the same type then* $a = b$*.*

2. Morphism *and* isomorphism *of incidence geometries are defined in an obvious way. Also for incidence geometries over a given type set* type preserving *morphism and isomorphism can be defined.*

The projective space $\mathcal{P}_V$ is an example of an incidence geometry with the type of a subspace being its dimension. Let $\mathcal{M}$ be the set of maximal p-locals of G. Let the type of a maximal p-local be its conjugacy class and define two maximal p-locals to be incidence if they contain a common Sylow p-subgroup of G. Then we have proved that $\mathcal{M}$ is isomorphic $\mathcal{P}_V$.

Let H be an arbitrary group and p prime. Let $\mathcal{M}(p, H)$ be the set of maximal p-locals of H and define two maximal p-locals to be incident if their intersection contains a Sylow p-subgroup of H.

Lemma 2.7 *$\mathcal{M}(p, H)$ is an incidence geometry if and only if $N_H(T) \leq M$ for each $M \in \mathcal{M}(p, H)$ and each $T \in \mathrm{Syl}_p(M)$.* MpHig

Proof: The incidence relation is clearly symmetric. It is reflexive if and only if each maximal p-local contains a Sylow p-subgroup of H. Also if M is a maximal p-local and $T \leq\in \mathrm{Syl}_p(M)$ with $N_H(T) \leq M$, then T is a Sylow p-subgroup of H. Indeed let $T \leq S \in Syl(H$ then $T \leq N_S(T) \leq M$ and so $N_S(T) = T$ and $S = T$. So we may assume from now on that each maximal$p$-local contains a Sylow p-subgroup of H.

Suppose now that $N_H(T) \not\leq M$ for some maximal p-local M and some Sylow p-subgroup T of M. Let $h \in N_H(T) \setminus M$. If $M = M^h$ then also $O_p(M) = O_p(M)^h$ and so $h \in N_H(O_p(M)) \leq M$, a contradiction. Thus $M \neq M^h$ and since $T \leq M \cap M^h$, M and M^h are incident and have the same type. Thus $\mathcal{M}(p, H)$ is not a incidence relation in this case.

Suppose now that $N_H(T) \leq M$ for each maximal p-local M and each Sylow p-subgroup T of M. Let $M_1, M_2 \in \mathcal{M}(p, H)$ be incident and of the same type. Then there exists a Sylow p-subgroup T of H and $h \in H$ with $T \leq M_1 \cap M_2$ and $M_1^h = M_2$. Then T and T^h

are both Sylow p-subgroups of M_2 and so $T = T^{hm}$ for some $m \in M_2$. Then $M_1^{hm} = M_2$ and $hm \in N_H(T) \leq M_1$. Hence $M_1 = M_2$ and $\mathcal{M}(p, H)$ is an incidence geometry. □

From the proof of the preceeding lemma we get

pig **Lemma 2.8** *Let H be a finite group, p a prime, S a Sylow p-subgroup of H and $B = N_G(S)$. Let*

$$\mathcal{B}(p, H) = \{P \leq H \mid B^h \leq P \text{ for some } h \in H\}.$$

Define two members of $\mathcal{B}(p, H)$ to be incident if their intersection is still in $\mathcal{B}(p, H)$ and let the type of $P \in \mathcal{B}(H)$ be P^H, the conjugacy class of P under H.

Then $\mathcal{P}(p, H)$ is an incidence geometry. More general, any subset of $\mathcal{B}(p, H)$ is an incidence geometry. □

The following lemma, which is an immediate consequence of 2.3a, c, gives a second group theoretic interpretation of the incidence in $\mathcal{M}$ for $G = GL_K(V)$.

ivqw **Lemma 2.9** *Let X and Y be proper subspaces of V. The X and Y are incident if and only if $O_p(N_G(X)) \leq N_G(Y)$.* □

Chapter 3

The p-local structure of $\mathrm{Sym}(\Omega)$

cpso

Let Ω be a finite set and p a prime. This chapter investigates the maximal p-locals of $G = \mathrm{Sym}(\Omega)$. For this let M be a maximal p-local subgroup of G, $Q = Q_M = O_p(M)$ and $X = X_M = \Omega_1 Z(Q_M)$. Then X is an elementary abelian normal p-subgroup of M and by the maximality of M, $M = N_G(X)$. Let $\Omega_1, \ldots, \Omega_k$ be the orbits for X on Ω with $|\Omega_1| > 1$. Since X is normal in M, each Ω_i is a block for M and M acts on $\{\Omega_1, \ldots, \Omega_k\}$. Choose notation so that $\{\Omega_1, \ldots, \Omega_r\}$ is the orbit of Ω_1 under M. For $1 \leq i \leq r$, let A_i be the image of X in $\mathrm{Sym}(\Omega_i)$ (viewed as a subgroup of $\mathrm{Sym}(\Omega)$). Let $A = A_1 A_2 \ldots A_k$. Then M normalizes A and as A is a non-trivial p-group, $M = N_G(A)$ by the maximality of M. Put $\Gamma = \Omega \setminus \bigcup_{i+1}^{r} \Omega_i$ and $B = A_1$. Then B is an elementary abelian p-subgroup acting regularly on Ω_1. It is easy to compute that $N_{\mathrm{Sym}(\Omega_1)}(B) \cong B \rtimes \mathrm{Aut}(B)$ and

$$M = N_G(A) \cong (B \rtimes \mathrm{Aut}(B)) \wr \mathrm{Sym}(r) \times \mathrm{Sym}(\Gamma)$$

Note that M is up to conjugation in $\mathrm{Sym}(\Omega)$ uniquely determined by $|B|$ and r (and $|\Gamma|$). We still need to check for which values of $|B|$ and r the resulting M is actually maximal. If not then M is properly contained in some maximal p-local M^*. Define Ω_i^*, B^*, A^* and Γ^* as above. Since M has only two orbits namely Γ and $\Omega \setminus \Gamma$ on Ω, M^* either has the same orbits or is transitive. This leads to the following three cases:

1. $\Gamma = \Gamma^*$

2. $\Gamma = \Omega \setminus \Gamma^*$

3. $\Gamma^* = \emptyset$ and $\Gamma \neq \emptyset$.

In the first case, Ω_1^* is block for M on $\Omega \setminus \Gamma$. Also we may assume that $\Omega_1^* \cap \Omega_1 \neq \emptyset$. Since $N_M(B)$ acts primitively on Ω_1 we conclude $\Omega_1 \subseteq \Omega_1^*$. As M is primitive on Ω_1^M, we conclude that $\Omega_1 = \Omega_1^*$ or $\Omega_1 \neq \Omega_1^* = \Omega \setminus \Gamma$. The first possibility gives $|M| = |M^*|$ and $M = M^*$ a contradiction. The second possibility leads to $B \rtimes \mathrm{Aut}(B) \leq GL(B^*)$ and

$B \rtimes \mathrm{Aut}(B)$ fixes all but $|B|$ vectors in B^*. But then $|B^*| - |B| \leq \frac{|B^*|}{p}$. Hence $|B^*| = |4|$ and $|B| = 2$. So $(|B|, r) = (2, 2)$ in this case.

In the second case $\mathrm{Sym}(\Gamma)$ and $\mathrm{Sym}(\Gamma^*)$ are both contained in M^* and so $M^* = \mathrm{Sym}(\Gamma) \times \mathrm{Sym}(\Gamma^*)$. This implies $(|\Gamma|, p) \in \{(2,2), (3,3), (4,2)\}$. Also $|\Omega^*| \neq |\Omega_1|$ or $|\Omega_1| > 4$, since otherwise $M = M^*$.

In the third case, we may assume that $\Gamma \cap \Omega_1^* \neq \emptyset$. Since $\mathrm{Sym}(\Gamma) \leq M \leq M^*$ and Ω_1^* is a block for M^*, we get $\Gamma \subseteq \Omega_1^*$. Hence $B^* \rtimes \mathrm{Aut}(B^*)$ is a primitive subgroup of $\mathrm{Sym}(\Gamma^*)$ containing a 2-cycle. Thus $B^* \rtimes \mathrm{Aut}(B^*) = \mathrm{Sym}(\Gamma^*)$ and $|B^*| \in \{2, 3, 4\}$. Since $C_M(\Gamma)$ is transitive on $\Omega \setminus \Gamma$ and also normalizes Ω_1^* we either have $\Omega_1^* = \Gamma$ or $\Omega_1^* = \Omega$.

We conclude that M is a maximal p-local maximal unless $(|\Gamma|, p) \in \{(2,2), (3,3), (4,2)\}$ or $(|B|, r) = (2, 2)$. In the exceptionell cases, M is not maximal unless $M = Sym(2) \times \mathrm{Sym}(4)$ in $\mathrm{Sym}(6)$.

It should be clear by now that the p-local structure of $\mathrm{Sym}(\Omega)$ is not nearly as nice as the one for $GL_K(V)$ in the right characteristic. Here we just list a view differences:

1. $\mathrm{Sym}(\Omega)$ usually has some maximal p-locals which do not contain a Sylow p-subgroup of G.
2. In $GL_K(V)$ all maximal p-locals are maximal subgroups, while this is usually not the case for $\mathrm{Sym}(\Omega)$.
3. $\mathrm{Sym}(\Omega)$ often has maximal p-locals with components. But in $PGL_K(V)$ the generalized Fitting subgroup of a p-local is always a p-group.

It might be worthwhile to determine those maximal p-locals which contain a Sylow p-subgroup. A necessary condition is that $B\,\mathrm{Aut}(B)$ contains a Sylow p-subgroup of $\mathrm{Sym}(\Omega_1)$. This is only the case for $|B| = p$ or $|B| = 4$. Let $|\Omega| = \sum t_i p^i$ with $0 \leq t_i < p$. Then for $|B| = p$, $|\Gamma|$ must be of the form $t_0 + \sum_{i>0} s_i p^i$ with $s_i \leq t_i$. For $|B| = 4$, $|\Gamma|$ has to be of the form $t_0 + 2t_1 + \sum_{i>1} s_i p^i$, $s_i \leq t_i$.

We illustrate to above calculation by determining the maximal 2-local subgroups of $\mathrm{Sym}(12)$:

1. $\mathrm{Sym}(2) \times \mathrm{Sym}(10)$ for $|B| = 2$ and $r = 1$,
2. $\mathrm{Sym}(4) \times \mathrm{Sym}(8)$ for $|B| = 4$ and $r = 1$, note that $\mathrm{Sym}(4) = 2^2 GL_2(2)$
3. $2^3 GL_3(2) \times \mathrm{Sym}(4)$ for $|B| = 8$ and $r = 1$, which is not maximal as it is contained in 2.
4. $Sym(2) \wr \mathrm{Sym}(2) \times \mathrm{Sym}(8)$ for $|B| = 2$ and $r = 2$, which is not maximal since it is contained in 2.
5. $\mathrm{Sym}(4) \wr \mathrm{Sym}(2) \times \mathrm{Sym}(4)$ for $|B| = 4$, $r = 2$, not maximal as it is contained in 2.
6. $\mathrm{Sym}(2) \wr \mathrm{Sym}(3) \times \mathrm{Sym}(6)$ for $|B| = 2$, $r = 3$.
7. $\mathrm{Sym}(4) \wr \mathrm{Sym}(3)$ for $|B| = 4$, $r = 3$.

8. $\mathrm{Sym}(2) \wr \mathrm{Sym}(4) \times \mathrm{Sym}(4)$ for $|B| = 2$, $r = 4$, not maximal as it is contained in 2.

9. $\mathrm{Sym}(2) \wr \mathrm{Sym}(5) \times \mathrm{Sym}(2)$ for $|B| = 2$, $r = 5$, not maximal as it is contained in 1.

10. $\mathrm{Sym}(2) \wr \mathrm{Sym}(6)$ for $|B| = 2$, $r = 6$.

Of these 2-locals 2., 4., 7. 8. and 10. contain a Sylow 2-subgroup, which has order 2^{10} while 1. 3. 5. 6. and 9. do not.

Chapter 4

A first look at groups of characteristic p-type

cfirstptype

In the proceeding chapters we investigated the p-local structure of $GL_n(p^k)$ and $\mathrm{Sym}(\Omega)$. Our general goal in theses lecture notes is to introduce techniques currently used to classify finite simple groups which are somewhat similar to $GL_n(p^k)$ in their p-local structure. For this purpose we define:

Definition 4.1 *Let G be a finite group and p a prime. Then we say that G is of* characteristic p-type *provided that the generalized Fitting subgroup of each p-local subgroup of G is a p-group.* dptype

It might be rather difficult to check whether a given group is of characteristic p-type, since this is a statement about all p-local subgroups of G. But it actually turns out that this can be verified just by looking at the maximal p-locals of G. Before we are able to prove this we will prove a few fundamental lemmas about action of p-groups on p'-groups.

Lemma 4.2 (Witt's Identity) *Let $a, b, c \in G$ then* WittID

$$[a, b^{-1}, c]^b [b, c^{-1}, a]^c [c, a^{-1}, b]^a = 1$$

Proof: Just write out the left hand side and cancel. □

Lemma 4.3 (Three Subgroup Lemma) *Let X, Y, Z be subgroups of group and $[X, Y, Z] = 1$ and $[Y, Z, X] = 1$, then $[Z, X, Y] = 1$.* TSL

Proof: This follows immediately from Witt's Identity. □

Lemma 4.4 *Let p be a prime, A be a finite group acting on the finite p-group Q.* qcqqa

(a) *If Q is abelian and A is a p'-group, then $Q = C_Q(A) \times [Q, A]$.*

(b) *If A is a p'-group, then $Q = C_Q(A)[Q, A]$ and $[Q, A] = [Q, A, A]$.*

(c) $[Q, O^p(A)] = [Q, O^p(A)] = [Q, O^p(A), k]$ *for all* $k \geq 2$.

(c) *If* $[Q, A, k] = 1$ *for some* $k \leq 2$*, then* $[Q, O^p(A)] = 1$.

Proof: (a) Let $a = |A|$ and $x \in Q$ put $m(x) = \sum_{b \in A} x^b$. Then m is an A-homomorphism from Q to $C_Q(A)$. Moreover, $m(x) = a \cdot x$ for all $x \in C_Q(A)$ and so m restricted to $C_Q(A)$ is an isomorphism. Thus $Q = C_Q(A) \oplus \ker m$. From this (or also directly from the defintion of m) we have $[Q, A] \leq \ker m$. If $m(x) = 0$ we get $-a \cdot x = \sum_{a \in A}(x^a - x)$ and so $x \in [Q, A]$. Thus $[Q, A] = \ker m$ and (a) holds.

(b) Let $\bar{Q} = Q/\Phi(Q)$. Then $\bar{Q} = C_{\bar{Q}}(A)[\bar{Q}, A]$. Let D be the inverse image of $C_{\bar{Q}}(A)$ in Q. Then $Q = D[Q, A]$. Suppose that $D < Q$. Then by induction $D = C_D(A)[D, A]$ and so $Q = C_D(A)[D, A][Q, A] = C_Q(A)[Q, A]$.

So we may assume that $D = Q$. Hence A acts trivially on $\bar{Q}$. If $\bar{Q}$ is cyclic, Q is cyclic and so abelian. So (b) follows from (a) in this case. Therefore we may assume that $\bar{Q}$ is not cyclic. Thus $\bar{Q} = \bar{D}_1\bar{D}_2$ for some proper subgroups $\bar{D}_i$ of $\bar{Q}$. Let D_i be the inverse image of $\bar{D}_i$ in Q. Since A acts trivially on $\bar{Q}$, D_i is A-invariant and so by induction, $D_i = C_{D_i}(A)[D_i, A]$. Thus $Q = D_1D_2 \leq C_Q(A)[Q, A]$ and the lemma is proved.

(c) Since $O^p(A)$ is generated by p' groups, (c) follows from the second statement in (b).

(d) follows directly from (c). □

AxB **Lemma 4.5 (Thompson's $A \times B$-Lemma)** *Let p be a prime, A a p-group and B a p' group. Suppose that $A \times B$ acts on the p-group Q such that B centralizes $C_Q(A)$. Then B centralizes Q.*

Proof: By induction on $|Q|$. Since $[Q, A] < Q$ we get $[Q, A, B] = 1$. Since $[A, B] = 1$ the "Three Subgroup Lemma" implies $[Q, B, A] = 1$. Thus $[Q, B] \leq C_Q(A)$ and $[Q, B, B] = 1$. As B is a p'-group, $[Q, B] = 1$ by4.4. □

f8pcop **Lemma 4.6** *Let M be a finite group and p a prime. Then $F^*(M)$ is a p-group if and only if $F^*(M) = O_p(M)$ and if and only if $C_M(O_p(M)) \leq O_p(M)$.*

Proof: Note that $O_p(M) \leq F(M) \leq F^*(M)$ and that $F^*(M)$ is a p-group if and only if $F^*(M0 \leq O_p(M)$. Hence the first two statements are equivalent. Suppose that $F^*(M) = O_p(M)$. Since $C_G(F^*(M)) \leq F^*(M)$ we conclude that $C_G(O_p(M)) \leq O_p(M)$.

Suppose next that $C_M(O_p(M)) \leq O_p(M)$. Since $E(M)$ and $O_{p'}(M)$ both centralize $O_p(M)$ we conclude that $E(M) = 1 = O_{p'}(M)$. Thus $F^*(M) = F(M)E(M) = O_p(M)O_{p'}(F(M)) = O_p(M)$ □

dgptype **Definition 4.7** *Let G be finite group and p a prime. A p-subgroup B of G is of* p-type *in G provided that $F^*(N_G(B))$ is a p-group.*

We remark that a finite group G is of characteristic p-type if and only all its non-trivial p-subgroups of p-type in G.

easyp-type **Lemma 4.8** *Let B be p-subgroup of G. Then the following statements are equivalent:*

(a) *$F^*(C_G(B))$ is a p-group.*

(b) *$C_G(BO_p(C_G(B)))$ is a p-group.*

(c) *$C_G(O_p(N_G(B))) \leq O_p(N_G(B))$.*

(d) *$F^*(N_G(B))$ is a p-group, that is B is of p-type in G.*

Proof: Put $Q^* = O_p(N_G(B))$ and $Q = O_p(C_G(B))$. Suppose (a) holds. Then $F^*(C_G(B)) = Q$, and as the generalized Fitting group contains its centralizer, $C_G(BQ) = C_{C_G(B)}(Q) \leq Q$. Thus (b) holds.

Suppose that (b) holds. Since $BQ \leq Q^*$, $C_G(Q^*) \leq C_G(BQ)$ and (c) holds by 4.6.

Also by 4.6 (c) implies (d).

Suppose that (d) holds. Let $E = O^p(C_{C_G(B)}(Q))$. By 4.6 in order to show that (a) holds, we need to verify that prove (a) we need to show that $C_{C_G(B)}(Q)$ is a p-group. But this is equivalent to showing that $E = 1$. Note that Q^* normalizes E and E normalizes Q^*. Thus $[Q^*, E] \leq Q^* \cap E \leq O_p(E)$. Since $O_p(E)$ is a normal p-subgroup of $C_G(B)$, $O_p(E) \leq Q$ and so $[O_p(E), E] = 1$ and $[Q^*, E, E] = 1$. Since $E = O^p(E)$, 4.4 implies $[Q^*, E] = 1$. By (d) $Q^* = F^*(N_G(B))$ and so $E \leq Q^*$ and $E = 1$. □

Lemma 4.9 *Let B be p-subgroup of G and $A \leq B$. If A is of p-type then B is of p-type.* ptype

Proof: Put $Q = O_p(C_G(A))$ and $T = O_p(C_G(B))$. Without loss $|B/A| = p$ and so B normalizes A. Let x be a p'-element in G centralizing BT. By 4.8b it suffices to show that $x = 1$. Since $C_G(B) \leq C_G(A)$, $C_Q(B)$ is a normal p-subgroup of $C_G(B)$ and so $C_Q(B) \leq T$. Hence $[C_Q(B), x] = 1$ and since B and x normalize Q, the $A \times B$-Lemma (4.5) implies, $[Q, x] = 1$. Since x centralizes A and A is of p-type, $x = 1$. □

Lemma 4.10 *Let M be a finite group with $F^*(M) = O_p(M)$. Then M is of characteristic p-type.* fpipt

Proof: Since $F^*(M) = O_p(M)$ the trivial group is of p-type. Hence by 4.9 all p-subgroups are of p-type and so by 4.8d, $F^*(L)$ is a p-group for all p-locals L of G. □

Corollary 4.11 *Let G be a group of characterictic p-type. Then each of the following groups are also of characteristic p-type:* ploptpt

(a) *p-local subgroups.*

(b) *Centralizers of non-trivial p-subgroups.*

(c) *Subgroups H of G such $O_p(H) \neq 1$ and H contains a Sylow p-subgroup of G.*

Proof: (a) and (b) follow immediately from 4.10 and **??**.

(c) Let $M = N_G(O_p(H))$ and S a Sylow p-subgroup of H. Then $S \leq M$ and since S is a Sylow p-subgroup of G and of M, $O_p(M) \leq S$. Thus $O_p(M) \leq H$ and $O_p(H) = O_p(M)$. Since $C_G(O_p(M) \leq O_p(M)$ we also have $C_H(O_p(H)) \leq O_p(H)$. Thus (c) holds by **??**.

Lemma 4.12 *Let G be a finite group and p a prime. Then the following are equivalent:* edtp

(a) *G is of characteristic p-type.*

(b) *All non-trivial p-subgroups of G are of p-type.*

(c) *All elements of order p in G are of p-type.*

(d) *$F^*(M) = O_p(M)$ for all maximal p-locals M of G.*

Proof: By defintion (a) implies (b). And obviously (b) implies (c). By 4.9 (c) implies (d). So suppose that (d) holds and let B be a non-trivial p-subgroup of G. Put $L = N_G(B)$ and let M be a maximal p-local subgroup of G containing L. Then $L = N_M(B)$ and so $F^*(L)$ is a p-group by 4.10. □

F*NG **Lemma 4.13** *Let G be a finite group and N a subnormal subgroup of G. Then $F^*(N) \leq F^*(G)$.*

Proof: This follows immediately from the definition of the generalized Fitting subgroup. □

cpp **Lemma 4.14** *$F^*(G) = O_p(G)$ if and only if G has no components and no non-trivial normal p'-subgroup.*

Proof: Suppose first that $F^*(G) = O_p(G)$ and let N be a non-trivial subnormal subgroup of G. Then $F^*(N) \leq F^*(G)$ and so $F^*(N)$ is a p-group. Hence N can neither be a component nor a p-group.

Conversely, if G has no components and no non-trivial normal p'-subgroups, then $O_{p'}(F(G)) = 1$, $F(G) = O_p(G)$ and $E(G) = 1$. Thus $F^*(G) = E(G)$. □

ngpt **Lemma 4.15** *Let G be a group of characteristic p-type.*

(a) *If $O_p(G) \neq 1$, then $F^*(N) = O_p(N)$ for each normal subgroup N of G.*

(b) *If $N \trianglelefteq G$, N is of characteristic p-type.*

Proof: (a) By 4.13 $F^*(N) \leq F^*(G) = O_p(G)$. Hence $F^*(N)$ is a p-group.

(b) Let D be a nontrivial p-subgroup of N. Then $N_N(D) \trianglelefteq N_G(D)$ and so (b) follows from (a) and 4.11. □

fpfa **Theorem 4.16 (Thompson)** *Let G be a finite group which admits a fixed-point-free automorphism of prime order. Then G is nilpotent.* □

gsgp **Lemma 4.17** *Let G be a group of characteristic p-type. Then one of the following holds:*

1. $F^*(G) = O_p(G)$.
2. $O_p(G) = 1$, $F^*(G) = F(G)$ *and all elements of order p in G act fixed-point-freely on* $F(G)$.
3. $F^*(G)$ *is simple and not a p'-group.*
4. G *is a p'-group.*

Proof: Suppose first that $O_p(G) \neq 1$. Then clearly G is a p-local of itself and $F^*(G) = O_p(G)$.

So suppose that $O_p(G) = 1$. Assume that N is a non-trivial normal p'-subgroup of G and let x be an element of order p in G. Then $C_N(x)$ is a normal p'-subgroup of $C_G(x)$ and so by 4.14 and 4.11b, $C_N(x) = 1$. Thus x acts fixed-point-freely on N and by 4.16 N is nilpotent.

Suppose that $F(G) \neq 1$. Then $F(G)$ is a p'-group. Let $D = C_G(F(G))$. Suppose that D is not a p'-group and let x be an element of order p in D. Then $F(G)$ is a non-trivial normal p' subgroup of $C_G(x)$, a contradiction. Hence D is a p'-group and so $D \leq F(G)$. Thus 2. holds in this case.

Suppose finally that $F(G) = 1$. Then also $O_{p'}(G) = 1$ and so none of the components of G are p' groups. Let E be a component of G. Then $Z(E) \leq F(G) = 1$ and E is simple. Suppose that L is a component of G different from E. Let $1 \neq x \in E$ with $|x| = p$. Then L is a component of $C_G(x)$ a contradiction, since x is of p-type and by 4.14. Thus $F^*(G) = E$. □

Definition 4.18 *Given a finite group G and a prime p. Then* dmhl

(a) $\mathcal{L}$ *is the set of p-locals of G.*

(b) $\mathcal{M}$ *is the set of maximal p-locals of G.*

(c) $\mathcal{H}$ *is the set of subgroups H of G with $F^*(H) = O_p(H)$.*

(d) *Let $\mathcal{T}$ be a set of subgroups of G and $H \leq G$. Then* $\mathcal{T}(H) = \{T \in \mathcal{T} \mid H \leq T\}$.

Chapter 5

Amalgams

Definition 5.1 *Let I be a set. Then an* amalgam *over I is a tuple* damalgam

$$\mathcal{A} = (P_J \mid J \subset I; \alpha_{J,K} \mid J \subset K \subset I),$$

where P_J is a group and $\alpha_{J,K} : P_J \mapsto P_K$ is a group homomorphism such that for all $J \subset K \subset L \subset I$, $\alpha_{J,K}\alpha_{K,L} = \alpha_{J,L}$.

The rank *of the amalgam is the cardinality of I.*

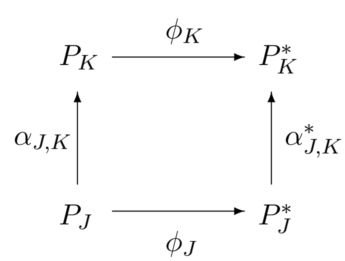
As an example let G a group and for $i \in I$ let P_i be subgroup of G. Put $P_J = \langle P_j \mid j \in J$ and for $J \subset K \subset I$ let $\alpha_{J,K} : P_J \mapsto P_K$ be the inclusion map.

For a second example let again G be a group and $G_i, i \in I$ a subgroup of G. For $J \subseteq I$ let $G_J = \bigcup_{j \in J} G_J$ and $P_J = G_{I \setminus J}$. Again let $\alpha_{J,K}$ be the inclusion map.

Definition 5.2 *Let $\mathcal{A}$ and $\mathcal{A}^*$ be amalgams over I. Then a morphism $\phi : \mathcal{A} \mapsto \mathcal{A}^*$ of* dmoram *amalgams over I is a tuple*

$$(\phi_J \mid J \subset I$$

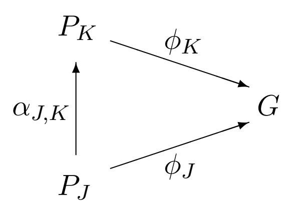
of group homomorphisms $\phi_J P_J \mapsto P_J$ such that for all $J \subset K \subset L \subset I$ the diagram



commmutes.

Definition 5.3 *Let $\mathcal{A} = (P_J \mid J \subset I; \alpha_{J,K} \mid J \subset K \subset I)$ be an amalgam of groups.* dcomp

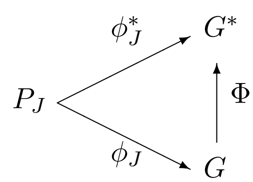
(a) *A* completion *of $\mathcal{A}$ is a tuple $(G; \phi_J \mid I)$ where G is a group, and $\phi_J : P_J \mapsto G$ is a homomorphism so that for all $J \subset K \subset I$ the diagram*



commutes and so that $G = \langle \phi_J(P_J) \mid J \subset I \rangle$.

(b) *The completion $(G; \phi_J \mid I)$ of $\mathcal{A}$ is called* faithful *if each of the ϕ_J is one to one.*

(c) *A completion $(G; \phi_J \mid I)$ of $\mathcal{A}$ is called* universal *provided that given any completion $(G^*; \phi_J^* \mid I)$ of $\mathcal{A}$ there exists a unique homomorphism $\Phi : G \mapsto G^*$ so that the diagram*



commutes.

euc **Lemma 5.4** *Let $\mathcal{A}$ be an amalgam. Then $\mathcal{A}$ has a (up to ismorphism) unique universal completion.*

Proof: Let G be the group with generators (g, J), where $J \subset I$ and $g \in P_J$, and relations $(g, J)(h, J) = (gh, J), J \subset I, g, h \in J$ and $(\alpha_{J,K}(g), K) = (g, J)$ for all $g \in P_J, J \subset K \subset I$. Define $\phi_J(g) = (g, J)$. Then is easy to check that $(G; \phi_J \mid I)$ is indeed a completion for $\mathcal{A}$, is universal and is unique up to isomorphism. □

If $|I| = 2$ and the α_J are one to one, it is not to difficult to see that the universal comletion is one to one. But for $|I| = 3$ the universal completion might be trivial as the following example shows.

Let $P_{1,2} = \mathrm{Sym}(3)$, $P_{2,3} = \mathrm{Alt}(4)$, $P_1 = P_3 = P_{1,3} = \mathrm{Alt}(3)$ $P_2 = \mathrm{Sym}(2)$ and $P_\emptyset = 1$. Choose the $\alpha_{J,K}$ arbitrary but one to one. Since $P_\emptyset = 1$ all the diagrams are clearly commutative and so this is indeed an amalgam. Let G be the universal completion and let

G_J be the image of P_J in G. Then $G_1 = G_3 = G_{1,3}$. Since $G_{1,2}$ is a quotient of Sym(3), $[G_1, G_2]$ ia a three group. Since $G_{2,3}$ is a quotient of Alt(4), $[G_1, G_2] = [G_3, G_2]$ is a 2-group. Thus $[G_1, G_2] = 1$. But in $G_{1,2}$ we see that $G_1 \le [G_1, G_2]$ and in $G_{2,3}$ that $G_2 \le [G_2, G_3]$. Thus $G_1 = G_2 = G_3 = 1$ and so also $G_{1,2} = G_{2,3} = G_{1,3} = 1$. But G is generated by the G_J's and so $G = 1$.

For a more encouraging example we will now prove:

Lemma 5.5 *Let K be a field, $G^* = GL_K(V)$ and V a n -dimensional vector space over K where $4 \le n < \infty$. Let $V_0 < V_1 < \ldots < V_n = V$ be a maximal flag in V and for $J \subset I = \{1, \ldots, n-1\}$ let $G^*_J = \bigcap_{j \in J} N_{G^*}(V_i)$. Then $(G^*, \phi^*_J, J \subset I)$ is the universal completion of the amalgam $(G^*_{I \setminus J} \mid J \subset I, \alpha_{J,K} \mid J \subset J \subset I\}$, where the ϕ^*_J's and $\alpha_{J,K}$ are the inclusion maps.* spg

Proof: Let $(G, \phi_J \mid J \subset I)$ be a universal completion of the amalgam and put $G_J = \phi_J(G^*_J)$. By definition of universal, there exists a homomorphism $\Phi : G \mapsto G^*$ with $\Phi \circ \phi_J = \phi^*_J$ for all $J \subset I$. In particular ϕ_J and $\Phi \mid_{G_J}$ are one to one. For $g \in G$, let $g^* = \Phi(g)$.

1. $G_J \cap G_K = G_J \cap \Phi^{-1}(G_K) = G_{J \cup K}$ spg-1

Note first that $G^*_J \cap G^*_K = G^*_{J \cap K}$ and so

$$G_{J \cap K} = \phi_{J \cap K}(G^*_J \cap G^*_K) = \phi_J(G^*_J \cap G^*_K) \le \phi_J(G_J)^* = G_J$$

So

$$G_{J \cap K} \le G_J \cap G_K \le G_J \cap \Phi^{-1}(G_K).$$

On the otherhand,

$$\Phi(G_J \cap \Phi^{-1}(G^*_K)) \le \Phi(G_J) \cap \Phi(\Phi^1(G^*_K)) \le G^*_J \cap G^*_K = G^*_{J \cup K} = \Phi(G_{J \cup K})$$

Thus $\Phi(G_J \cap G_K) = \Phi(G_J \cap \Phi^{-1}(G_K)) = \Phi(G_{J \cap K})$ and as Φ is one on G_J, 1. is proved.

2. Let $h \in G_K$ with $G^*_J \cap G^{*h^*}_J \le G^*_K$. Then beginitemize spg-2

(a) $G_J \cap G^h_J = G_{J \cup K} \cap G^h_{J \cup K}$.

(b) Φ maps $G_J \cap G^h_J$ isomorphicly onto $G^*_J \cap G^{*h^*}_J$.

Since $\Phi(G_J \cap G^h_J) \le G^*_J \cap G^{*h^*}_J \le G^*_K$ we conclude that

$$G_J \cap G^h_J = G_J \cap G^h_J \cap \Phi^{-1}(G_K) = (G_J \cap \Phi^{-1}(G_K)) \cap (G_J \cap \Phi^{-1}(G_K))^h$$

So by 1.

$$G_J \cap G^h_J = G_{J \cup K}) \cap G_{J \cup K})^h.$$

Thus (a) holds. (b) follows from (a) since Φ restricted to G_J.

Let $v_1, v_2 \dots v_n$ be a basis for V with $v_i \in V_i \setminus V_n$. Let $1 \leq i < j \leq n$ and define $h^*_{ij} \in G^*$ by $v_i \mapsto v_j, v_j \mapsto v_i$ and $v_k \mapsto v_k$ for $k \neq i, j$. Then $h^*_{ij} \in G^*_k$ for all $k \neq i$ and $h_{ij} \not G^*_i$. Put $h_{ij} = \phi_{I-j}(h^*_{ij}$. Then by 1.

spg-3 **3.** $h_{ij} \in G_k$ for all $k \neq i$, $h_{ij} \notin G_i$ and $h^2_{ij} = 1$.

Let $h = h_{12}$, $\Gamma = G/G_1$ and view Γ has a graph with edges $\{(G_1g, G_1hg)g \mid \in G\}$. Also note that G acts on Γ by right multiplication. The main goal now is to show that Γ is a complete graph, i.e that any two distint vertices are adjacent. This amounts to showing that Γ is connected and that every neighbor of G_1 (other than G_1h) is also a neighbor of G_1h. For this the will investigate paths of length 1 and 2 in Γ

spg-4 **4.** Γ is undirected and connected.

Since $h^2 = 1$, $(G_1h, G_1) = (G_1h, G_1hh)$ is an edge. Also any other edge is conjugate to (G_1, G_1h) and so (x, y) is an edge if and only if (y, x) is an edge. Next let Γ_0 be the connected component of Γ containg G_1. Then Γ_0 is a block for G on Γ. As G_1 fixes $G_1 \in \Gamma_0$. G_1 normalizes Γ_0. Also both G_1 and G_1h are in Γ_0 and thus h fixes Γ_0.

Let $i > 1$. Since G^*_i acts doubly transitively on the 1-spaces in V_i, $G^*_{1i} = G^*_1 \cap G^*_i$ is a maximal subgroup of G^*_i. Hence also G_{1i} is a maximal subgroup of G_i. Also $h \in G_i \setminus G_{1i}$ and so $G_i = \langle G_{1i}, h\rangle$. Thus G_i normalizes Γ_0. Since G is a completion, $G = \langle G_i \mid i \in I\rangle$. Therefore G normalize Γ_0. But G acts transitively on Γ, and so $\Gamma = \Gamma_0$.

spg-5 **5.** Let $\Delta(G_1)$ be the set of neighbors of G_1 in Γ.

(a) $\Delta_1 = \{G_1hg \mid g \in G_1\}$.

(b) G_1 acts transitively on Δ_1.

(c) Via Φ the action of G_1 on Δ_1 is isomorphic to the action of G^*_1 on the 1-spaces in $V \setminus V_1$.

Let $G_1x \in \Gamma$ so that G_1x is adjacent to G_1. Then $(G_1, G_1x) = (G_1g, G_1hg)$ for some $g \in G$ and so $g \in G_1$ and $G_1x = G_1xg$. Since $V_2 = V_1 + V_1^{h^*}$, $G^*_1 \cap G^{*h^*}_1 \leq G^*_2$. Henc (a) and (b) hold.

Note that $G_1h \in \Delta(G_1)$ and the stabilzer of G_1h in G_1 is $G_1 \cap G^h_1$. By 2. $\Phi(G_1 \cap G^h_1) = G^*_1 \cap G^{*h^*}_1$. Now $G^*_1 \cap G^{*h^*}_1$ is the stabilzer of the 1-space $V_1^{h^*}$ in G^*_1 and so also (c) is proved.

spg-6 **6.** G has three orbits on paths of length two in Γ. Representatives can by chosen as follows:

1. (G_1h, G_1, G_1h)

2. (G_1h, G_1, G_1t) where $t \in G_{I\setminus 1}$ with $t^* : v_1 \mapsto v_1 + v_2$ and $v_i \mapsto v_i$ for all $i \geq 2$.

3. (G_1h, G_1, G_1h_{13}).

Let (a, b, c) be a path of length two. Since G is transitive on the edges we may assume that $a = G_1h$ and $b = G_1$. Hence there is a one to one correspondce between G orbits of paths of length two and orbits of $G_1 \cap G_1^h$ on $\Delta(G_1)$. And so by 5.c, we need to determine the orbits of $G_1^* \cap G_1^{*h^*}$ on the 1-spaces in $V \setminus V_1$. But these orbits clearly consist of $V_1^{h^*}$, the 1-spaces in $V_2 \setminus \{V_1, V_1^{h^*}\}$ and the 1-spaces. Representatives are given by $V_1^{h^*} = Kv_2, V_1^{t^*} = K(v_1 + v_2)$ and $V_1^{h_{13}} = Kv_3$. Thus 6. holds.

7. Let $\Lambda = \{G_1g \mid g \in G_3\}$ viewed as a subgraph of Γ. Then Λ is complete. spg-7

Clearly G_3 acts transitively on Λ. The stablizer of the vertex G_1 in G_3 is $G_1 \cap G_3$. By 1. we conclude that via Φ the action of G_3 on Λ is isomomorphic to the action of G_3^* and $G_3^*/G_1^* \cap G_3^8$, which in turn is just the action of G_3^* on the 1-spaces in V_3. This latter action is double transitive. So Λ is either complete or has no edges. Since (G_1, G_1h) is an edge in Λ, Λ is complete.

8. Γ is complete. spg-8

Since Γ is connected, it suffices to prove that for any path (a, b, c) in Γ with $a \neq c$ we have that a and c are adjacent. By 6. we just need to show that G_1h is adjacent to G_1t and G_1h_{13}. But h, t, h_{13} are all in G_3 and so G_1h, G_1t, G_1h_{13} are in Λ. So they are adjacent by 7..

9. $G = G_1 \cup G_1hG_1$. spg-9

By 8. $\Gamma = \{G_1\} \cup \Delta(G_1)$. So 9. follows from 5.a.

We are now able to finish the proof of 5.5 by showing that Φ is an isomorphism. Since G^* is generated by the $G_i^* = \Phi(G_i)$, Φ is onto. Let $g \in G$ with $\Phi(g) = 1$. If $g \in G_1$ then $g = 1$, since Φ restricted to G_1 is one to one. If $g \notin G_1$, then by 9., $g = xhy$ for some $x, y \in G_1$. Hence $1 = \Phi(x)h^*\Phi(y)$ and $h_{12}^* = h^* = \Phi(x)^{-1}\Phi(y)^{-1} \in G_1^*$, a contradiction to 3.. □

Chapter 6

The Amalgam Method

cam

Definition 6.1 *Let G be a finite group, $T \leq G$ and p a prime.* dpclosed

(a) *G is* p-closed *provided that G has a unique Sylow p-subgroup.*

(b) *G is* T-minimal *if $T \not\trianglelefteq G$ and T lies in a unique maximal subgroup of G.*

(c) *G is* T-connected *if $T \not\trianglelefteq G$ and for each normal subgroup N of H either $N \cap T$ is normal in H or $H = NT$.*

(d) *G is* p-minimal *if G is S-minimal for some Sylow p-subgroup S of G, i.e. G is not p-closed and a Sylow p-subgroup of G lies in a unique maximal subgroup of G.*

(e) *G is* p-connected *if G is S-connected for some Sylow p-subgroup S of G, i.e. G is not p-closed and if N is a normal subgroup of G, then either N is p-closed or G/N is a p-group.*

We remark that G is p-closed if and only if $O_p(G)$ is a Sylow p-subgroup of G and if and only if $G/O_p(G)$ is p'-group.

Lemma 6.2 *Let G be a finite group, p a prime and suppose that G is p-minimal. Let $S \in Syl_p(G)$, let M be the unique maximal subgroup of G containing S and $E = \mathrm{Core}_M(G)$ and $L = O^p(G)$ Then* pmpc

(a) *G is p-connected.*

(b) *E is p-closed.*

(c) *LE/E is the unique minimal normal subgroup of G/E.*

(d) *$E/O_p(G) = \Phi(G/E)$ is nilpotent.*

(e) *Suppose that G is solvable. Then*

(a) *LE/E is an elementary abelian q-group for some prime $q \neq p$.*

(b) *$L/O_p(L)$ is a q-group .*

(c) *G is a $\{p,q\}$-group.*

(d) *S acts irreducible on LE/E.*

Proof: (a) Let $N \trianglelefteq G$. Then $G = N_G(S \cap N)N$. Hence not both $N_G(S \cap N)$ and NS can be contained in M. Hence $S \cap N \trianglelefteq G$ or $G = NS$.

(b) Since $ES \leq M$, $ES \neq M$ and so since G is p-connected E is p-closed.

(c) Let R/E be a non-trivial normal subgroup of G/E. Then $R \trianglelefteq G$ and so $R \not\leq M$ and $G = RS$. Thus $L \leq R$.

(d) Let $R/O_p(G)$ be a maximal subgroup of $G/O_p(G)$ and suppose that $E \not\leq R$. Then $G = ER$. Then $|G/R| = |ER/R| = |E/E \cap R|$ divides $E/O_p(E)$ and so R contains a Sylow p-subgroup of G. Thus $S^g \leq R$ for some g and so $R \leq M^g$ and $G = ER \leq M^g$, a contradiction. Hence R/E is contained in all the maximal subgroups of G/E and so $R/O_p(G) \leq \Phi(G/O_p(G)$. Conversely M/E is a maximal subgroup of G/E and so $\Phi(G/O_p(G)) \leq M/O_p(G)$. As $\Phi(G/O_p(G)) \trianglelefteq G/O_p(G)$, $\Phi(G/O_p(G) \leq E/O_p(G)$.

(e) Since G is solvable, (d) implies that LE/E is an elementary abelian q-group for some prime q. As $L = O^p(L)$, $q \neq p$. So (ea) holds. Let $R/O_p(L)$ be a Sylow q-subgroup of $L/O_p(L)$. Then $G = N_G(R)L$ and $LE = RE$. Thus $G = N_G(R)E$. Note also that $[O_p(G), R] \leq [O_p(G), L] \leq O_p(G) \cap L \leq O_p(L) \leq R$ and so $O_p(G) \leq N_G(R)$. Thus by (d) $N_G(R)/O_p(G) = G/O_p(G)$. Hence RS is a normal subgroup of G and as $R \not\leq M$, $G = RS$. Thus $L \leq R$ and $L = R$ and (eb) holds. Inparticular, L and so also $G = LS$ are $\{p,q\}$ groups and (ec) is proved. For (ed) let T/E be an non-trivial S invariant subgroup of LE/E. Since LE/E is abelian, T/E is normal in LE/E. Since $G = LS$ we conclude that T/E is normal in G/E and so by (c), $T/E = LE/E$. □

Of particular importance to us will be certain kind of rank 2 amalgams with given faithful completion G.

dapt **Definition 6.3** *Let p be a prime. An* amalgam of p-type *is a tuple (G, G_1, G_2, B) such that*

(a) *G is a group, G_1 and G_2 are finite subgroups of G, $B = G_1 \cap G_2$ and $G = \langle G_1, G_2 \rangle$*

(b) *$F^*(G_i) = O_p(G_i)$ for $i = 1, 2$.*

(c) *No non-trivial subgroup of B is normal in both G_1 and G_2.*

(d) *$O_p(G_i) \leq B$ for $i = 1, 2$.*

(e) *Let $S \in \mathrm{Syl}_p(B)$ and put $Z_i = \langle \Omega_1 Z(S)^{G_i} \rangle$. Then for each $i \in \{1, 2\}$ one of the following holds.*

1. *$C_{G_i}(Z_i)$ is p-closed*
2. *Put $V_i = \langle Z_{3-i}^{G_i} \rangle$. Then $\Omega_1 Z(S) \trianglelefteq G_i$ and $C_{G_i}(V_i)$ is p-closed.*

bapt **Example 6.4** *Let G be a group of characteristic p-type and S a Sylow p-subgroup of G. Suppose that S lies in at least two different maximal p-locals of G. Then there exists $M^*, L \in \mathcal{H}(S)$ so that $(\langle M^*, L\rangle, M^*, L, M^* \cap L))$ is an amalgam of p-type.*

Proof:

1. Let $S \leq H \leq G$ with $O_p(H) \neq 1$. Then $F^*(H) = O_p(H)$. bapt-1

Put $L = N_G(O_p(H))$. Then $S \in \mathrm{Syl}_p(L)$ and so $O_p(L) \leq S \leq H$. Thus $O_p(L) \leq O_p(H)$ and so $O_p(H) = O_p(L)$. Since G is of characteristic p-type, $C_H(O_p(H)) \leq C_G(O_p(L) \leq O_p(L)$. Thus 1. holds.

Let $H \in \mathcal{H}$ and define $Z_H = \langle \Omega_1 Z(T) \mid T \in \mathrm{Syl}_p(H)\rangle$ and $C_H = C_H(Z_H)$.

2. $1 \neq Z_H \leq \Omega_1 Z(O_p(H))$. bapt-2

Let $T \in \mathrm{Syl}_p(H)$. Then $O_p(H) \leq T$ and so $Z(T) \leq C_H(O_p(H)) \leq O_p(H)$. Thus $Z(T) \leq Z(O_p(H))$. This immediately implies 2.

Note that for $H \in \mathcal{H}(S)$, $Z_H = \langle \Omega_1 Z(S)^H\rangle$. Pick $M \in \mathcal{M}(S)$ with Z_M maximal and put $M^* = N_M(S \cap C_M)$.

3. bapt-3

(a) $Z_{M^*} = Z_M$

(b) $O_p(M^*) = S \cap M$

(c) C_{M^*} is p-closed.

(d) $\mathcal{M}(M^*) = \{M\}$, that is M is the unique maximal p-local of G containg M^*.

By Frattini, $M = C_M M^*$ and so

$$Z_M = \langle \Omega_1 Z(S)^M\rangle = \langle \Omega_1 Z(S)^{C_M M^*}\rangle = \langle \Omega_1 Z(S)^{M^*}\rangle = Z_{M^*}$$

Thus (a) holds. By (a) $C_{M^*} = C_M \cap M^*$. Since $S \cap C_M \leq C_M \cap M^*$ and $S \cap C_M$ is a Sylow p-subgroup of C_M, $S \cap C_M$ is a Sylow p-subgroup of $C_M \cap M^* = C_{M^*}$. Also $S \cap C_M$ is normal in M^* and so (c) holds. By 2., $O_p(M^*) \leq C_M^*$ and so $O_p(M^*) = O_p(C_{M^*}) = S \cap C_M$. Hence (b) holds.

For (d) let $M^* \leq H \in \mathcal{M}$. Then

$$Z_M = Z_{M^*} = \langle \Omega_1 Z(S)^{M^*}\rangle \leq \langle \Omega_1 Z(S)^H\rangle == Z_H.$$

So $Z_M \leq Z_H$ and the maximality of Z_M implies $Z_M = Z_H$. Thus $H \leq N_G(Z_M) = M$ and $H = M$

By assumptions there exists maximal p-local containing S whic is different from M. In particular, there exists $L \in \mathcal{H}(S)$ with $L \not\leq M$. Choose such an L which is minimal with respect to inclusion.

4. Let $D \leq M^* \cap L$ so that D is normal in M^* and L. Then $D = 1$. bapt-4

Suppose that $O_p(D) \neq 1$. Then $M^* \leq N_G(O_p(D)))$ and so by 3.d, $N_G(O_p(D)) \leq M$, a contradiction since L normalizes $O_p(D))$. Thus $O_p(D) = 1$. Since $D \trianglelefteq M$ and $F^*(M) = O_p(M)$, 4.13 implies $F^*(D) = O_p(D) = 1$. Thus $D = 1$.

bapt-5 **5.** $M \cap L$ is the unique maximal subgroup of L containg S. In particular, L is p-minimal or p-closed.

Let $S \leq P < H$. Then $O_p(L) \leq S \leq P$ and so by 1. $P \in \mathcal{H}(S)$. Hence by minimality of L, $P \leq M$ and 4. is established.

bapt-6 **6.** One of the following holds

1. C_L is p-closed.
2. $\Omega_1(Z(S) \leq Z(L)$ and $C_L(V)$ is p-closed, where $V = \langle Z_M^L \rangle$

We may assume that C_L is not p-closed. Since subgroups of p-closed groups are p-closed, L is not p-closed. So by 5. L is p-minimal. Thus by 6.2 $L = C_L S$. Thus $\Omega_1(Z(S) \leq Z(L)$. Suppose that $C_L(V)$ is not p-closed. Then $L = C_L(V)S$ and so Z_M is normalized by $L = C_L S$. Hence $L \leq N_G(Z_M) \leq Z_M$. This proves 6.. □

Its now readily verified that $(\langle M^*, L\rangle, M^*, L, M^* \cap L)$ meets all the parts of the definiton of an amalgam of p-type. □

cdg **Definition 6.5** *Let G be a group and $\{G_i, i \in I)$ a non-empty tuple of subgroups.*

(a) *The* coset graph $\Gamma(G; G_i, i \in I)$ *of G with respect to G_i, $i \in I$ is the graph with vertices $\{(K,i) \mid i \in I, K \in G/G_i\}$ (that is the disjoint union of the $G/G_i, i \in I$) and two vertices (K,i) and (L,j) are adjacent if and only if $i \neq j$ and $K \cap L \neq \emptyset$.*

(b) *G acts Γ by $(K,i)^g = (Kg, i)$.*

When working with coset graphs we will usually abuse notation and just write K for (K,i)

ccg **Lemma 6.6** *Let G be a group and $G_i, i \in I$ subgroups of G. Put $\Gamma = \Gamma(G; G_i, i \in I)$*

(a) *The orbits of G on Γ are exactly the G/G_i's for $i \in I$.*

(b) *Each edge in Γ is conjugate to edge of the form (G_i, G_j), $i \neq j \in I$.*

(c) *Let $H = \langle G_i \mid \in I$. Let Λ be the connected component of Γ containing one (and so all) of the $G_i's$. Then $H = N_G(\Lambda)$ and $\Lambda = \Gamma(H; G_i, i \in I$. In particular, Γ is connected if and only if $G = \langle G_i \mid i \in I \rangle$.*

(d) *$C_G(\Gamma)$ is exactly the largest normal subgroup of G contained in all the $G_i, i \in I$.*

Proof: (a) Note that $|\Lambda$ is block for G on Γ. Since G_i fixes the vertex G_i of Γ.

(b) Let (G_ig, G_jh) be an edge. Then there exists $k \in G_ig \cap G_jh$ and so $(G_ig, G_jh) = (G_ik, G_jk)(G_i, G_j)^k$. As G_ik and G_jk are distinct in Γ, $i \neq j$. So (b) holds.

(c) Clear each G_i normalizes Λ and so $H \leq N_G(\Lambda)$. Let $\Xi = \Gamma(H; G_i, i \in I)$. Then each vertex in Ξ is conjugate under H to some G_i and so $\Xi \subset \Lambda$.

Let $G_ig \in \Gamma$ so that G_ig is adjacent to some $G_jh \in \Xi$. The there exists $k \in G_ig \cap G_jh$. Hence $k \in G_jh \subseteq H$ and so $G_ig = G_ik \in \Xi$. It follows that Ξ is a union of connected components of Γ and so $\Xi = \Lambda$.

Let $g \in N_G(\Lambda)$, then $G_ig \in \Lambda = \Xi$ and so $g \in H$ and (a) is established.

(d) This is readily verified. □

Hypothesis 6.7 *(G, G_1, G_2, B) is an amalgam of p-type and $\Gamma = \Gamma(G; G_1, G_2)$.*

Lemma 6.8 edtr

(a) *Γ is connected.*

(b) *G acts faithfully on Γ.*

(c) *Each edge in Γ is conjugate to (G_1, G_2).*

(d) *Each vertex in Γ is conjugate to exactly on of G_1 and G_2.*

Proof: (a) By part (a) of the definition 6.3, $G = \langle G_1, G_2\rangle$. So Γ is connected by 6.6a.

(b) Let $N = C_G(\Gamma)$. Then $N \trianglelefteq G$ and $N \leq G_1 \cap G_2 = B$. Thus by part (c) of the defintion of an amalgam of p-type, $N = 1$.

(c) and (d) follow immediately from 6.4a,b. □

The *amalgam method* uses the graph Γ to define normal subgrous of the G_i and then tries to derive information of the actions of G_i on these normal subgroup. The next definition gives names to the most frequently used normal subgroups.

Definition 6.9 *Let $\gamma, \delta \in \Gamma$ be adjacent vertices in Γ and i a non-negative integer.* dsam

(a) *G_δ is the stabilizer of δ in G and so $G_\delta = G_i^g$ if $\delta = G_ig$.*

(b) *If Λ is a subset of Γ then* $G_\Lambda = \bigcap_{\lambda\in\Lambda} G_\lambda$.

(c) $d(\gamma, \delta)$ *is the length of a shortest path from γ to δ in Γ.*

(d) $\triangle^i(\delta) = \{\gamma \in \Gamma \mid d(\gamma, \delta) \leq i\}$. $\triangle(\delta) = \triangle^1(\delta) \setminus \{\delta\}$ *is the set of neigbors of δ in Γ*

(e) $G^i_\delta = G_{\triangle^i(\delta)}$

(f) $Q_\delta = O_p(G_\delta)$.

(g) $Z_{\delta\gamma} = \langle \Omega_1 Z(T) \mid T \in \mathrm{Syl}_p(G_{\delta\gamma})\rangle$.

(h) $Z_\delta = \langle Z_{\delta\gamma} \mid \gamma \in \triangle(\delta)\rangle$.

(i) $X_\delta = \Omega_1 Z(Q_\delta)$.

(j) $V^i_\delta = \langle Z_\gamma \mid \gamma \in \triangle^i(\delta)\rangle$, $V_\delta = V^1_\delta$ *and* $W_\delta = V^2_\delta$.

We remark the Z_{G_i} defined in example 6.4 is the same has the Z_{G_i} defined here.

The parameter b defined in the next lemma is of fundamental importance to the amalgam method:

db **Definition 6.10**

(a) *For* $\delta \in \Gamma$ *let* $b_\delta = \min\{d(\delta,\rho) \mid \rho \in \Gamma, Z_\delta \notin G^1_\rho\}$

(b) $b = \min_{\delta\in\Gamma} = \min\{b_{G_1}, b_{G_2}\}$

(c) *A* critical pair *is a pair of vertices* (α,α') *in* Γ *with* $Z_\alpha \not\leq G^1_{\alpha'}$ *and* $d(\alpha,\alpha') = b$.

From now an (α,α') will always denoted a critical pair, (δ,γ) is an arbitrary edges in Γ,

$$(\alpha,\alpha+1,\alpha+2,\ldots,\alpha+b-1,\alpha+b) = (\alpha'-b,\alpha'-b+1,\ldots,\alpha'-1,\alpha')$$

is a shortest path from α to α' and $\beta = \alpha+1$.

bza **Lemma 6.11**

(a) $F^*(G_\delta) = Q_\delta$.

(b) $Z_\delta \leq X_\delta \leq Q_\delta \leq G^1_\delta \leq G_\gamma$.

(c) *b is a positive integer.*

(d) *Let* $\rho \in \Gamma$ *with* $d(\delta,\rho) \leq b_\delta$. *Then* $Z_\delta \leq G_\rho$.

(e) $Z_\alpha \leq G_{\alpha,\alpha+1,\alpha+2,\ldots,\alpha'-1,\alpha'}$

(f) $Q_\delta \cap G^1_\rho \leq Q_\gamma$.

(g) *Let* $\rho \in \Gamma$ *with* $d(\delta,\rho) < b_\delta$. *Then* $Z_\delta \leq Q_\rho$.

(h) *One of the following holds:*

1. $C_{G_\delta}(Z_\delta)$ *is p-closed with* Q_δ *its unique Sylow p-subgroup.*
2. $Z_\delta = Z_{\delta\gamma} \leq Z_\gamma$ *and* $C_{G_\delta}(V_\delta)$ *is p-closed with its Sylow p-subgroup contained in* Q_δ.

Moreover, if 2. holds for δ, *then 1. holds for* γ.

Proof: (a) Since G_δ is conjugate to G_1 or G_2 this follows from part (a) of the definition 6.3.

(b) Using (a), $Z_\delta \leq Q_\delta$ is proved just as in 2.. By part (d) of the definition 6.3. and since $\{\delta, \gamma\}$ is conjugate to $\{G_1, G_2\}$, $Q_\delta \leq G_{\delta\gamma}$. Since γ is an arbitrary neighbor of δ, $Q_\delta \leq G^1_\delta$. $G^1_\delta \leq G_\gamma$ holds by the definition of G^1_δ.

(c) The first statement in (b) just says that $b_\delta \neq 0$. If $b = \infty$, then since Γ is connected, Z_δ fixes all the vertices in Γ, a contradiction to the faithful action of G on Γ.

(d) Let λ be a neighbor of ρ with $d(\delta, \lambda) < d(\delta, \rho)$. Then $d(\delta, \rho) < b_\delta$ and so $Z_\delta \leq G^1_\lambda \leq G_\rho$.

(e) follows from (d)

(f) Since $G^1_\gamma \leq G_\delta$, G^1_γ normalizes $Q_\delta \cap G^1_g$. Hence $O_\delta \cap G^1_g$ is a normal p-subgroup of G^1_g. Thus $Q_d \cap G^1_\gamma \leq O_p(G^1_\gamma) \leq O_p(G_\gamma) = Q_\gamma$.

(g) By induction on $d(\delta, \rho)$. If $\delta = \rho$ this holds by (b). So pick λ as in (d). Then by induction, $Z_\delta \leq Q_\lambda$. By definition of b_δ, $Z_\delta \leq G^1_\rho$. Hence by (f) $Z_\delta \leq Q_\lambda \cap G^1_\rho \leq Q_\rho$.

(h) Suppose that $C_{G_\delta}(Z_\delta)$ is not p-closed. Then by part (e) of 6.3, $\Omega_1 Z(S) \trianglelefteq G_\delta$ and $C_{G_\delta}(V_\delta)$ is p-closed, where S is a Sylow p-subgroup of $G_{\gamma\delta}$. Hence $Z_\delta = \Omega_1 Z(S) = Z_{\delta\gamma} \leq Z_\gamma$. So 2. holds. If 2. holds for both δ and γ, then $Z_\delta = Z_{\gamma\delta} = Z_\gamma$ is normal in G_γ and G_δ, a contradiction to 6.3c. □

Definition 6.12 *Let G be finite group, p a prime and V a $GF(p)G$-module.* dff

(a) *V is called an* "Failure of Factorization-module *(or* FF-module *) for G provided that there exists an subgroup A of G so that*

  (a) *A is an elementary abelain p-group with $[V, A] \neq 1$.*

  (b) *$|V/C_V(A)| \leq |A|$.*

(b) *Let A be as in (a). Then we say that A is* offending *on V or that A is an* offender *on V.*

Lemma 6.13 (a) *$Z_\alpha \leq G_{\alpha'}$ and $Z_{\alpha'} \leq G_\alpha$.*

(b) *$C_{G_\alpha}(Z_\alpha)$ is p-closed.*

(c) *Suppose that $[Z_\alpha, Z_{\alpha'}] \neq 1$. Then Z_α is offending on $Z_{\alpha'}$ or $Z_{\alpha'}$ is offending on Z_α. Moreover, (α', α) is a critical pair.*

(d) *Suppose that $[Z_\alpha, Z_{\alpha'}] = 1$. Then $C_{G_\beta}(Z_\beta)$ is not p-closed, b is odd, $Z_\beta = Z_{\alpha\beta} \leq Z_\alpha$ and (α', α) is not a critical pair.*

(e) *One of the following holds:*

  1. *b is even, $[Z_\alpha, Z_{\alpha'}] \neq 1$ and Z_α is an FF-module for G_α.*
  2. *b is odd, $[Z_\alpha, Z_{\alpha'}] \neq 1$ and one of Z_α and Z_β is an FF-module for G_α and G_β, respectively.*

3. *b is odd, $[Z_\alpha, Z_{\alpha'}] = 1$ and $Z_\beta = Z_{\alpha\beta} \leq Z_\alpha$.*

Proof: (a) This follows from 6.11d.

(b) Otherwise 6.11 implies that $Z_\alpha \leq Z_\beta$. But then $Z_\beta \not\leq G^1_{\alpha'}$, a contradiction the the minimality of b.

(c) Note that either $|Z_\alpha/C_{Z_\alpha}(Z_{\alpha'})| \leq |Z_{\alpha'}/C_{Z_{\alpha'}}(Z_\alpha)|$ or $|Z_{\alpha'}/C_{Z_{\alpha'}}(Z_\alpha)| \leq |Z_\alpha/C_{Z_\alpha}(Z_{\alpha'})|$. In the first case Z_α is an offender on $Z_{\alpha'}$ and in the second $Z_{\alpha'}$ is an offender on Z_α. Suppose that $Z_{\alpha'} \leq G^1_\alpha$. Then by 6.11f, $Z_{\alpha'} \leq Q_\beta \cap G^1_\alpha \leq Q_\alpha$ 6.11, $[Z_\alpha, Z_{\alpha'}] = 1$, a contradiction.

(d) Suppose that $C_{G_{\alpha'}}(Z_{\alpha'})$ is p-closed. Then as Z_α is a p-group centralizing $Z_{\alpha'}$, $Z_\alpha \leq Q_{\alpha'} \leq G^1_{\alpha'}$, a contradiction. Therefore $C_{G_{\alpha'}}(Z_{\alpha'})$ is not p-closed. Thus by (b) α and α' are not conjugate in Γ and so b is odd. In particular, β is conjugate to α' and so the remaining parts of (d) follow from 6.11(h).

(e) This just rephrases (c) and (d). □

We next lemma illustrate that the restricted nature of FF-modules.

FFS3 **Lemma 6.14** *Let $p = 2$ and V be a faithful FF-module for $H = \mathrm{Sym}(3)$. Then $V = [V, G] \oplus C_V(G)$ and $|[V, H]| = 4$.*

Proof: Let $A \leq H$ so that A meets the definition of an FF-module. Then $|A| = 2$ and so $1 \neq V/C_V(A) \leq |A| = 2$. Thus $|V/C_V(A)| = 2$. Let B be a conjugate of A with $B \neq A$. Then $H = \langle A, B\rangle$ and so

$$C_V(A) \cap C_V(B) = C_V(H)$$

Since both $C_V(A)$ and $C_V(B)$ have index 2 in V, we conclude that $|V/C_V(A)| \leq 4$. Note that $H' = Alt(3) \cong C_3$. By Maschke, $V = C_V(H') \oplus [V, H']$. Also $|[V, H']| \geq 4$ and $[V, H' \cap C_V(H) = 0$. We conclude that $V = [V, H'] \oplus C_V(H)$ and $[V, H'] = [V, H]$ has order 4. □

fodg **Lemma 6.15** *G acts faithfully on δ^G.*

Proof: Let $D = C_G(\delta^G)$. Then D is a normal subgroup of G_δ and so $F^*(D) = O_p(D) \leq O_p(G_\delta) \leq G_\gamma$. Hence $F^*(D)$ is a normal subgroup of G contained in $G_{\delta\gamma}$ and so $F^*(D) = 1$. Therefore also $D = 1$. □

trans **Lemma 6.16** *For $\rho \in \{\delta, \gamma\}$ let $H_\rho \leq G_\rho$ so that H_ρ acts transitively on $\triangle(\rho)$. Then no non-trivial subgroup of G_δ is normalized by H_δ and H_γ.*

Proof: Let $H = \langle H_\delta, H_\gamma$. The $H_\rho \leq H \cap G_\rho$ and $H \cap G_\rho$ acts transitively on $\triangle(\rho)$. Let $\Lambda = \delta^H \cup \gamma^H$. We claim that $\Lambda = \Gamma$. For this let $\nu \in \Gamma$ be adjacent to $\lambda \in \Lambda$. Since λ is conjugate to δ or γ under H, $H \cap G_\lambda$ acts transitively on $\triangle(\lambda)$ and some neighbor say μ of λ lies in Λ. Thus $\nu \in \mu^{H \cap G_\lambda} \subseteq \Lambda$. As Γ is connected, $\Lambda = \Gamma$.

Let $D \leq G_\delta$ so that D is normalized by H_δ and H_γ. Then H normalizes G and so D acts trivially on $\delta^H = \delta^G$. Thus by 6.15, $D = 1$. □

Chapter 7

(Sym(3),Sym(3))-Amalgams

cs3s3

As an introduction to some of the arguments used in the amalgam method we will now investigate amalgams of 2-type such that $G_0/O_2(G_i) \cong \mathrm{Sym}(3)$ for $i = 1, 2$. Such amalgam had been first considered by D. Goldschmid [Go] and indeed it was that article there the amalgam method had been introduced. Since then the amalgam methode had been refined and modified, most notable by B. Stellmacher for example in [DGS]. The arguments given here basicly follow [DGS], allthough our treatment is less elegant since I deliberately avoided the use of the parameter s, where s is roughly the smallest postive integer so the G acts transitively on paths of length s with conjugate starting points. This make the proof less geometric, but has the advantage that the arguments apply to a broader class of groups.

Hypothesis 7.1 hs3s3

(a) (G, G_1, G_2, B) *is an amalgam of* 2*-type.*

(b) *For both* $i = 1$ *and* $i = 2$, $G_i/O_2(G_i) \cong \mathrm{Sym}(3)$ *or* $\mathrm{Alt}(3)$.

We also will use the notation introduced in chapter 6.

Lemma 7.2 gabtg

(a) $G_{\delta\gamma}$ *is a 2-group.*

(b) *If* $Q_\delta \not\leq Q_\gamma$, *then* $G_{\delta\gamma} = Q_\delta\, Q_\gamma$ *is a Sylow* 2*-subgroup of* G_γ *and* $G_\gamma/Q_\gamma \cong \mathrm{Sym}(3)$

(c) $Q_\beta \not\leq Q_\alpha$, $G_{\alpha\beta} = Q_\alpha\, Q_\beta$ *is a Sylow* 2*-subgroup of* G_α *and* $G_\alpha/Q_\alpha \cong \mathrm{Sym}(3)$.

(d)
1. $G_{\alpha\beta} = Q_\alpha Q_\beta$ *is a Sylow* 2*-subgroup of* G_α *and* G_β *and both* G_α/Q_α *and* G_β/G_β *are isomorphic to* $\mathrm{Sym}(3)$
2. $Q_\alpha \leq Q_\beta = G_{\alpha\beta}$, Q_β *is a Sylow* 2*-subgroup of* G_α, $G_\alpha/Q_\alpha \cong \mathrm{Sym}(3)$ *and* b *is even.*

Proof: (a) Suppose that $G_{\delta\gamma}$ is not a 2-group. Since $Q_\delta \leq G_{\delta\gamma}$ and $|G_\delta/Q_\delta|$ divides six, $|G_\delta/G_{\delta\gamma}| \leq 2$. Thus $G_{\delta\gamma}$ is normal in G_δ and G_γ. We conclude that $G_{\delta\gamma} = 1$, a contradiction. Hence the first statement holds.

(b) Suppose that $Q_\delta \not\leq Q_\gamma$. Since $|G_{\delta\gamma}/Q_\gamma| \leq 2$, we conclude that this number is two and $G_{\delta\gamma} = Q_\delta \mathrm{Q}_\gamma$. So (b) holds.

(c) Suppose that $Q_\beta \leq Q_\alpha$. Then $Q_\beta < Q_a \leq G_{\alpha\beta}$. Thus $Q_\alpha = G_{\alpha\beta}$ and so $Z_\alpha \leq X_\alpha = \Omega_1 Z(G_{\alpha\beta}) \leq Z_\beta$. But then $Z_\beta \not\leq G^1_{\alpha'}$, a contradiction to the minimality of b.

(d) If $Q_\alpha \not\leq Q_\beta$, then (d1) holds by (a) and (b)

Suppose that $Q_\alpha \leq Q_\beta$. Then $Q_\alpha < Q_\beta \leq G_{\alpha\beta}$ and so $Q_\beta = G_{\alpha\beta}$ and $|G_{\alpha\beta}/Q_\alpha| = 2$. If b is odd, then $(\alpha'-1, \alpha')$ is conjugate to (α, β) and so $Z_\alpha \leq Q_{\alpha'-1} \leq Q_{\alpha'}$, a contradiction. Thus (d2) holds in this case. □

Rn1s3 **Lemma 7.3** *Suppose that* $[Z_\alpha, Z_{\alpha'}] \neq 1$. *Then*

(a) $Z_\alpha Q_{\alpha'} = G_{\alpha'\alpha'-1}$.

(b) *Both* Z_α *and* $Z_{\alpha'}$ *are* FF*-modules for* G_α *and* $G_{\alpha'}$ *respectively.*

Proof: (a) This follows from $|G_{\alpha'-1\alpha'}/Q_{\alpha'}| = 2$, $Z_\alpha \leq G_{\alpha'-1\alpha'}$ and $Z_\alpha \not\leq Q_{\alpha'}$.

(b) This follows since

$$|Z_\alpha/C_{Z_a}(Z_{\alpha'})| \leq |Z_\alpha/Z_\alpha \cap Q_{\alpha'}| = |Z_\alpha Q_{\alpha'}/Q_{\alpha'}| \leq |G_{\alpha'-1\alpha'}/Q_{\alpha'}| = 2 \leq |Z_{\alpha'}/C_{Z_\alpha}(Z_\alpha)|.$$

□

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