STATE SURFACES OF LINKS

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1. Overview

State surfaces are spanning surfaces of links that are obtained from link diagrams. Their construction is guided by the combinatorics underlying Kauffman's construction of the Jones link polynomial via *state models*. Geometric properties of state surfaces are often dictated by simple link diagrammatic criteria, and the surfaces themselves carry important information about geometric structures of link complements. On the other hand, certain state surfaces carry spines (state graphs) that can be used to compute the Jones polynomial of links. From this point of view, state surfaces provide a tool for establishing relations between Jones polynomials and topological link invariants, such as the crosscap number or invariants coming from geometric structures on link complements (e.g. hyperbolic volume). In this article we survey the construction of state surfaces of links and some of their recent applications.

2. Definitions and examples

For a link K in S^3, $D = D(K)$ will denote a link diagram, in the equatorial 2–sphere of S^3. We will often abuse by referring to the projection 2–sphere using the common term projection plane. In particular, $D(K)$ cuts the projection "plane" into compact regions each of which is a polygon with vertices at the crossings of D.

Given a crossing on a link diagram $D(K)$ there are two ways to resolve it; the A- resolution and the B-resolution as shown in Figure 2. The figure is borrowed from [13]. Note that if the link K is oriented, only one of the two resolutions at each crossing will respect the orientation of K. A Kauffman state σ on $D(K)$ is a choice of one of these two resolutions at each crossing of $D(K)$ [14]. For each state σ of a link diagram the *state graph* $\mathbb{G}_\sigma$ is constructed as follows: The result of applying σ to $D(K)$ is a collection $v_\sigma(D)$ of non-intersecting circles in the plane, called *state circles*, together with embedded arcs recording the crossing splice. Next we obtain the *state surface* S_σ, as follows: Each circle of $v_\sigma(D)$ bounds a disk in S^3. This collection of disks can be disjointly embedded in the ball below the projection plane. At each crossing of $D(K)$, we connect the pair of neighboring disks by a half-twisted band to construct a surface $S_\sigma \subset S^3$ whose boundary is K.

Figure 1. The A-resolution (left), the B-resolution (right) of a crossing and their contribution to state surfaces.

Example 2.1. Given an oriented link diagram $D = D(K)$, the Seifert state, denoted by $s(D)$, is the one that assigns to each crossing of D the resolution that is consistent with the orientation of D. The corresponding state surface $S_s = S_s(D)$ is oriented (a.k.a. a Seifert surface). The process of constructing S_s is known a *Seifert's algorithm* [17].

Key words and phrases. State surface, state graph, Jones polynomial, crosscap number, hyperbolic volume.

The author is supported in part by NSF grants DMS-1404754 and DMS-1708249.

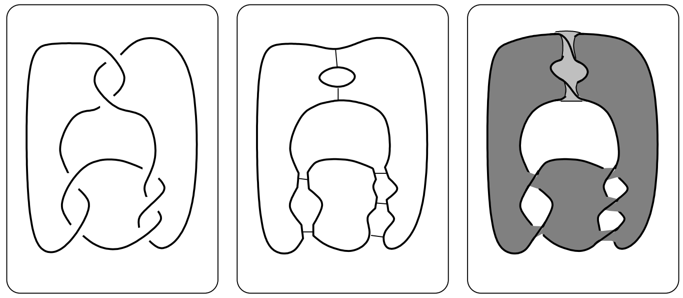

FIGURE 2. Left to right: A diagram, the all-A state graph $\mathbb{G}_A$ and the corresponding state surface S_A.

By applying the A–resolution to each crossing of D, we obtain a crossing–free diagram $s_A(D)$. Its state graph, denoted by $\mathbb{G}_A = \mathbb{G}_A(D)$, is called the all–$A$ state graph and the corresponding state surface is denoted by $S_A = S_A(D)$. An example is shown in Figure 2, which is borrowed from [10]. Similarly, for the all–B state the crossing–free resulting diagram is denoted by $s_B(D)$, the state graph is denoted $\mathbb{G}_B$, and the state surface by S_B.

By construction, $\mathbb{G}_\sigma$ has one vertex for every circle of v_σ (i.e. for every disk in S_σ), and one edge for every half–twisted band in S_σ. This gives a natural embedding of $\mathbb{G}_\sigma$ into the surface, where vertices are embedded into the corresponding disks, and edges run through the corresponding half-twisted bands. Hence, $\mathbb{G}_\sigma$ is a spine for S_σ.

Lemma 2.2. *The surface S_σ is orientable if and only if $\mathbb{G}_\sigma$ is a bipartite graph.*

Proof. Recall that a graph is bipartite if and only if all cycles (i.e. paths from any vertex to itself) contain an even number of edges.

If $\mathbb{G}_\sigma$ is bipartite, we may assign an orientation on S_σ, as follows: Pick a normal direction to one disk, corresponding to a vertex of $\mathbb{G}_\sigma$, extend over half–twisted bands to orient every adjacent disk, and continue inductively. This inductive process S_σ will not run into a contradiction since every cycle in $\mathbb{G}_\sigma$ has even number of edges. Thus S_σ is a two–sided surface in S^3, hence orientable. This is the case with the example of Figure 2.

Conversely, suppose $\mathbb{G}_\sigma$ is not bipartite, hence contains a cycle with an odd number of edges. By embedding $\mathbb{G}_\sigma$ as a spine of S_σ, as above, we see that this cycle is an orientation–reversing loop in S_σ. □

□

3. GENUS AND CROSSCAP NUMBER OF ALTERNATING LINKS

The genus of an orientable surface S with with k boundary components is defined to be $1-(\chi(S)+k)/2$, where $\chi(S)$ is the Euler characteristic of S and the *crosscap number* of a non-orientable surface with k boundary components is defined to be $2 - \chi(S) - k$.

Definition 3.1. Every link in S^3 bounds both orientable and non-orientable surfaces. The *genus* of an oriented link K, denoted by $g(K)$, is the minimum genus over all orientable surfaces S bounded by K. That is we have $\partial S = K$. The *crosscap number* (a.k.a. non-orientable genus) of a link K, denoted by $C(K)$, is the minimum crosscap number over all non-orientable surfaces spanned by K.

For *alternating links* the genus and the crosscap number can be computed using state surfaces of alternating link diagrams. For the orientable case, we recall the following classical result due to Crowell [7] (see also [17]).

Theorem 3.2. [7] *Suppose that D is a connected alternating diagram of a k-component link K. Then the state surface $S_s(D)$ corresponding to the Seifert state of D realizes the genus of K. That is we have $g(K) = 1 - (\chi(S_s(D)) + k)/2$.*

In [2], Adams and Kindred used state surfaces to give an algorithm for computing crosscap numbers of alternating links. To summarize their algorithm and state their result, consider a connected alternating diagram $D(K)$ as 4-valent a graph on S^2. Each region in the complement of the graph is an m-gon with vertices at the vertices of the graph.

Lemma 3.3. *Suppose that $D(K)$ is a connected alternating link diagram whose complement has no bigons or 1-gons. Then at least one region must be a triangle.*

Proof. Let V, E, F denote the number of vertices, edges and complimentary regions of $D(K)$, respectively. Then, $V - E + F = 2$ and $E = 2V$, which implies that $F > V$. Suppose that none of the F regions is a triangle. Then, $F < 4V/4 = V$ since each region has at least four vertices and each vertex can only be on at most 4 distinct regions. This is a contradiction. □ □

Observe that the Euler characteristic of a surface, corresponding to a state σ, is $\chi(S_\sigma) = v_\sigma - c$, where c is the number of crossings on $D(K)$. Thus to maximize $\chi(S_\sigma)$ we must maximize the number of state circles v_σ. Now we outline the algorithm from [2] that finds a surface of maximal Euler characteristic (and thus of minimum genus) over all surfaces (orientable and non-orientable) spanned by an alternating link.

Adams-Kindred algorithm: Let $D(K)$ be a connected, alternating diagram.

(1) Find the smallest m for which the complement of the projection $D(K)$ contains an m-gon.
(2) If $m = 1$, then we resolve the corresponding crossing so that the 1-gon becomes a state circle.
Suppose that $m = 2$. Then some regions of $D(K)$ are bigons. Create one branch of the algorithm for each bigon on $D(K)$. Resolve the two crossings corresponding to the vertices of the bigon so that the bigon is bounded by a state circle. See Figures 1.4 and 5 below.
(3) Suppose $m > 2$. Then by Lemma 3.3, we have $m = 3$. Pick a triangle region on $D(K)$. Now the process has two branches: For one branch we resolve each crossing on the triangle's boundary so that the triangle becomes a state circle. For the other branch, we resolve each of the crossings the opposite way.

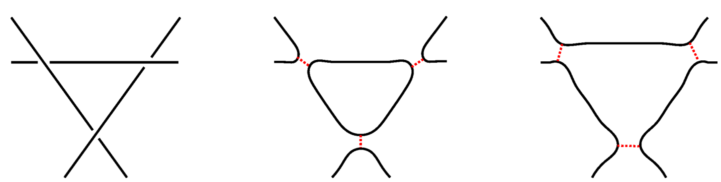

FIGURE 3. The two branch of the algorithm for triangle regions. The is figure borrowed from [13].

(4) Repeat Steps 1 and 2 until each branch reaches a projection without crossings. Each branch corresponds to a Kauffman state of $D(K)$ for which there is a corresponding state surface. Of all the branches involved in the process choose one that has the largest number of state circles. The surface S corresponding to this state has maximal Euler characteristic over all the states corresponding to $D(K)$. Note that, *a priori*, more than one branches of the algorithm may lead to surfaces of maximal Euler characteristic.

Theorem 3.4. [2] *Let S be any maximal Euler characteristic surface obtained via above algorithm from an alternating diagram of k-component link K. Then,*

(1) If there is a surface S as above that is non-orientable then $C(K) = 2 - \chi(S) - k$.

(2) If all the surfaces S as above are orientable, we have $C(K) = 3 - \chi(S) - k$. Furthermore, S is a minimal genus Seifert surface of K and $C(K) = 2g(K) + 1$.

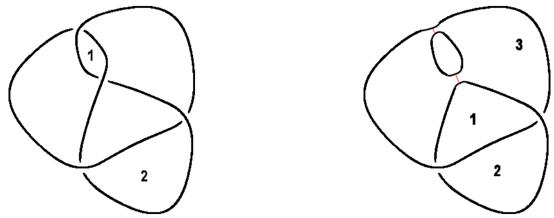

FIGURE 4. A diagram of 4_1 with bigon regions 1 and 2 and the result of applying step 2 of the algorithm to bigon 1.

Example 3.5. Different choices of branches as well as the order in resolving bigon regions following the algorithm above, may result in different state surfaces. In particular at the end of the algorithm we may have both orientable and non-orientable surfaces that share the same Euler characteristic:

Suppose that we choose the bigon labeled by 1 in the left hand side picture of Figure 1.4. Then, for the next step of the algorithm, we have three choices of bigon regions to resolve, labeled by 1 and 2 and 3 of the figure.

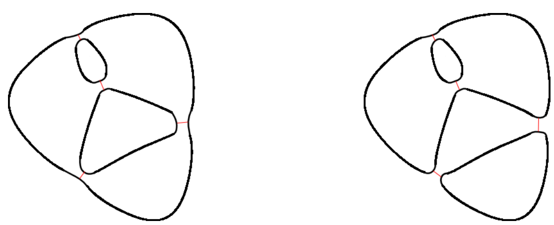

FIGURE 5. Two algorithm branches corresponding to different bigons.

The choice of bigon 1 leads to a non-orientable surface, shown in the left panel of Figure 5, realizing the crosscap number of 4_1, which is two. The choice of bigon 2 leads to an orientable surface, shown in the right panel of Figure 5, realizing the genus of the knot which is one. Both surfaces realize the maximal Euler characteristic of -1.

4. JONES POLYNOMIAL AND STATE GRAPHS

A connected link diagram D defines a 4–valent planar graph $\Gamma \subset S^2$, which leads to the construction of the *Turaev surface* $F(D)$ as follows [8]: Thicken the projection plane to $S^2 \times [-1, 1]$, so that Γ lies in $S^2 \times \{0\}$. Outside a neighborhood of the vertices (crossings) the surface intersect $S^2 \times [-1, 1]$, in $\Gamma \times [-1, 1]$. In the neighborhood of each vertex, we insert a saddle, positioned so that the boundary circles on $S^2 \times \{1\}$ are the components of the A–resolution $s_A(D)$, and the boundary circles on $S^2 \times \{-1\}$ are the components of $s_B(D)$.

When D is an alternating diagram, each circle of $s_A(D)$ or $s_B(D)$ follows the boundary of a region in the projection plane. Thus, for alternating diagrams, the surface $F(D)$ is the projection sphere S^2. For general diagrams, the diagram D still is alternating on $F(D)$.

The surface $F(D)$ has a natural cellulation: the 1–skeleton is the graph Γ and the 2–cells correspond to circles of $s_A(D)$ or $s_B(D)$, hence to vertices of $\mathbb{G}_A$ or $\mathbb{G}_B$. These 2–cells admit a checkerboard coloring, in which the regions corresponding to the vertices of $\mathbb{G}_A$ are white and the regions corresponding to $\mathbb{G}_B$ are shaded. The graph $\mathbb{G}_A$ (resp. $\mathbb{G}_B$) can be embedded in $F(D)$ as the adjacency graph of white (resp. shaded) regions. The *faces* of $\mathbb{G}_A$ (that is, regions in the complement of $\mathbb{G}_A$) correspond to vertices of $\mathbb{G}_B$, and vice versa. Hence the graphs are dual to one another on $F(D)$. Graphs, together with such embeddings into an orientable surface, called *ribbon graphs* have been studied in the literature [4]. Building on this point of view, Dasbach, Futer, Kalfagianni, Lin and Stoltzfus [8] showed that the ribbon graph embedding of $\mathbb{G}_A$ into the Turaev surface $F(D)$ carries at least as much information as the Jones polynomial $J_K(t)$.

To state the relevant result from [8], recall that a *spanning* subgraph of $\mathbb{G}_A$ is a subgraph that contains all the vertices of $\mathbb{G}_A$. Given a spanning subgraph $\mathbb{G}$ of $\mathbb{G}_A$ we will use $v(\mathbb{G})$, $e(\mathbb{G})$ and $f(\mathbb{G})$ to denote the number of vertices, edges and faces of $\mathbb{G}$ respectively.

Theorem 4.1. [8] *For a connected link diagram D, the Kauffman bracket $\langle D\rangle \in \mathbb{Z}[A, A^{-1}]$ is expressed as*

$$\langle D\rangle = \sum_{\mathbb{G}\subset\mathbb{G}_A} A^{e(\mathbb{G}_A)-2e(\mathbb{G})}(-A^2 - A^{-2})^{f(\mathbb{G})-1},$$

where $\mathbb{G}$ ranges over all the spanning subgraphs of $\mathbb{G}_A$.

Given a diagram $D = D(K)$, the *Jones polynomial* of K, denoted by $J_K(t)$, is obtained from $\langle D\rangle$ as follows: Multiply $\langle D\rangle$ by $(-A)^{-3w(D)}$, where $w(D)$ is the *writhe* of D, and then substitute $A = t^{-1/4}$ [14, 17].

Theorem 4.1 leads to formulae for the coefficients of $J_K(t)$ in terms of topological quantities of the state graphs $\mathbb{G}_A$, $\mathbb{G}_B$ corresponding to any diagram of K [8, 9]. These formulae become particularly effective if $\mathbb{G}_A, \mathbb{G}_B$ contain no 1-edges loops. In particular, this is the case when $\mathbb{G}_A, \mathbb{G}_B$ correspond to an alternating diagram that is *reduced* (i.e. contains no redundant crossings).

Corollary 4.2. [9] *Let $D(K)$ be a reduced alternating diagram and let β_K and β'_K denote the second and penultimate coefficient of $J_K(t)$, respectively . Let $\mathbb{G}'_A$ and $\mathbb{G}'_B$ denote the* simple *graphs obtained by removing all duplicate edges between pairs of vertices of $\mathbb{G}_A(D)$ and $\mathbb{G}_B(D)$. Then,*

$$|\beta_K| = 1 - \chi(\mathbb{G}'_B), \quad \text{and} \quad |\beta'_K| = 1 - \chi(\mathbb{G}'_A).$$

5. Geometric Connections

To a link K in S^3 corresponds a compact 3-manifold with boundary; namely $M_K = S^3 \smallsetminus N(K)$, where $N(K)$ is an open tube around K. The interior of M_K is homeomorphic to the link complement $S^3 \smallsetminus K$. In the 80's, Thurston [19] proved that link complements decompose canonically into pieces that admit locally homogeneous geometric structures. A very common and interesting case is when the entire $S^3 \smallsetminus K$ has a hyperbolic structure, that is a metric of constant curvature -1 of finite volume. By Mostow rigidity, this hyperbolic structure is unique up to isometry, hence invariants of the metric of $S^3 \smallsetminus K$ give topological invariants of K.

State surfaces obtained from link diagrams $D(K)$ give rise to properly embedded surfaces in M_K. Many geometric properties of state surfaces can be checked through combinatorial and link diagrammatic criteria. For instance, Ozawa [18] showed that the all -A surface $S_A(D)$ is π_1–injective in M_K if the state graph $\mathbb{G}_A(D)$ contains no 1-edge loops. Futer, Kalfagianni and Purcell [10] gave a different proof of Ozawa's result and also showed that M_K is a fiber bundle over the circle with fiber $S_A(D)$, if and only if the simple state graph $\mathbb{G}'_A(D)$ is a *tree.*

State surfaces have been used to obtain relations between combinatorial or Jones type link invariants and geometric invariants of link complements. Below we give a couple of sample of such relations. For additional applications the reader is referred to to [1, 5, 10, 11, 15, 16] and references therein. The first result, proven combining [2] with hyperbolic geometry techniques, relates the crosscap number and the Jones polynomial of alternating links. It was used to determine the crosscap numbers of 283 alternating knots of *knot tables* that were previously unknown [6].

Theorem 5.1. [13] *Given an an alternating, non-torus knot K, with crosscap number $C(K)$, we have*

$$\left\lceil \frac{T_K}{3} \right\rceil + 1 \;\leq\; C(K) \;\leq\; \min\left\{ T_K + 1,\ \left\lfloor \frac{s_K}{2} \right\rfloor \right\}$$

where $T_K := |\beta_K| + |\beta'_K|$, β_K, β'_K are second and penultimate coefficients of $J_K(t)$ and s_K is the degree span of $J_K(t)$. Furthermore, both bounds are sharp.

Example 5.2. For $K = 4_1$ we have $J_K(t) = t^{-2} - t^{-1} + 1 - t + t^2$. Thus $T_K = 1$ and $s_k = 4$ and Theorem 5.1 gives $C(K) = 2$.

The next result gives a strong connection of the Jones polynomial to hyperbolic geometry as it estimates volume of hyperbolic alternating links in terms of coefficients of their Jones polynomials. The result follows by work of Dasbach and Lin [9] and work of Lackenby [15].

Theorem 5.3. *Let K be an alternating link whose exterior admits a hyperbolic structure with volume* $\mathrm{vol}(S^3 \smallsetminus K)$. *Then we have*

$$\frac{v_{\mathrm{oct}}}{2}(T_K - 2) \leq \mathrm{vol}(S^3 \smallsetminus K) \leq 10 v_{\mathrm{tet}}(T_K - 1),$$

where $v_{\mathrm{oct}} = 3.6638$ *and* $v_{\mathrm{tet}} = 1.0149$.

To establish the lower bound of Theorem 5.3 one looks at the state surfaces S_A, S_B corresponding to a reduced alternating diagram $D(K)$: Use $M_K \backslash\backslash S_A$ to denote the complement in M_K of a collar neighborhood of S_A. Jaco-Shalen-Johannson theory [12] implies that there is a canonical way to decompose $M_K \backslash\backslash S_A$ along certain annuli into three types of pieces: (i) I–bundles over subsurfaces of S_A; (ii) solid tori; and (iii) the remaining pieces, denoted by $\mathrm{guts}(M, S)$. On one hand, by work Agol, Storm, and Thurston [3], the quantity $|\chi(\mathrm{guts}(M_K, S_A))|$ gives a lower bound for the volume $\mathrm{vol}(S^3 \smallsetminus K)$. On the other hand, [15] shows that this quantity is equal to $1 - \chi(\mathbb{G}'_A)$, which by Corollary 4.2 is $|\beta'_K|$. A similar consideration applies to the surface S_B giving the lower bound of Theorem 5.3. The approach was developed and generalized to non-alternating links in [10].

References

1. C. Adams, A. Colestock, J. Fowler, W. Gillam, and E. Katerman, *Cusp size bounds from singular surfaces in hyperbolic 3-manifolds*, Trans. Amer. Math. Soc. **358** (2006), no. 2, 727–741. 5
2. Colin Adams and Thomas Kindred, *A classification of spanning surfaces for alternating links*, Algebr. Geom. Topol. **13** (2013), no. 5, 2967–3007. 3, 5
3. Ian Agol, Peter A. Storm, and William P. Thurston, *Lower bounds on volumes of hyperbolic Haken 3-manifolds*, J. Amer. Math. Soc. **20** (2007), no. 4, 1053–1077, with an appendix by Nathan Dunfield. 6
4. Béla Bollobás and Oliver Riordan, *A polynomial invariant of graphs on orientable surfaces*, Proc. London Math. Soc. (3) **83** (2001), no. 3, 513–531. 4
5. Stephan D. Burton and Efstratia Kalfagianni, *Geometric estimates from spanning surfaces*, Bull. Lond. Math. Soc. **49** (2017), no. 4, 694–708. 5
6. Jae Choon Cha and Charles Livingston, *Knotinfo: Table of knot invariants*, `http://www.indiana.edu/~knotinfo`, June 14 2018. 5
7. Richard Crowell, *Genus of alternating link types*, Ann. of Math. (2) **69** (1959), 258–275. 2
8. Oliver T. Dasbach, David Futer, Efstratia Kalfagianni, Xiao-Song Lin, and Neal W. Stoltzfus, *The Jones polynomial and graphs on surfaces*, J. Combin. Theory Ser. B **98** (2008), no. 2, 384–399. 4, 5
9. Oliver T. Dasbach and Xiao-Song Lin, *On the head and the tail of the colored Jones polynomial*, Compos. Math. **142** (2006), no. 5, 1332–1342. 5, 6
10. David Futer, Efstratia Kalfagianni, and Jessica Purcell, *Guts of surfaces and the colored Jones polynomial*, Lecture Notes in Mathematics, vol. 2069, Springer, Heidelberg, 2013. 2, 5, 6
11. David Futer, Efstratia Kalfagianni, and Jessica S. Purcell, *Quasifuchsian state surfaces*, Trans. Amer. Math. Soc. **366** (2014), no. 8, 4323–4343. 5
12. William H. Jaco and Peter B. Shalen, *Seifert fibered spaces in* 3*-manifolds*, Mem. Amer. Math. Soc. **21** (1979), no. 220, viii+192. 6
13. Efstratia Kalfagianni and Christine Ruey Shan Lee, *Crosscap numbers and the Jones polynomial*, Adv. Math. **286** (2016), 308–337. 1, 3, 5
14. Louis H. Kauffman, *State models and the Jones polynomial*, Topology **26** (1987), no. 3, 395–407. 1, 5
15. Marc Lackenby, *The volume of hyperbolic alternating link complements*, Proc. London Math. Soc. (3) **88** (2004), no. 1, 204–224, With an appendix by Ian Agol and Dylan Thurston. 5, 6
16. Marc Lackenby and Jessica S. Purcell, *Cusp volumes of alternating knots*, Geom. Topol. **20** (2016), no. 4, 2053–2078. 5
17. W. B. Raymond Lickorish, *An introduction to knot theory*, Graduate Texts in Mathematics, vol. 175, Springer-Verlag, New York, 1997. 1, 2, 5
18. Makoto Ozawa, *Essential state surfaces for knots and links*, J. Aust. Math. Soc. **91** (2011), no. 3, 391–404. 5
19. William P. Thurston, *The geometry and topology of three-manifolds*, Princeton Univ. Math. Dept. Notes, 1979. 5

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