Rank 3 Latin square designs

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Abstract

A Latin square design whose automorphism group is transitive of rank at most 3 on points must come from the multiplication table of an elementary abelian p-group, for some prime p.

1 Introduction

We are interested in the classification of all proper partial linear spaces that admit a group of automorphisms acting with rank 3 on points. Significant progress has been made when the group is assumed to be primitive [5]. Here we consider the most basic imprimitive case.

A *transversal design* $TD(k,n)$ is a partial linear space $(\mathcal{P},\mathcal{L})$ in which $\mathcal{P} = \cup_{i=1}^{k} B_i$ is the disjoint union of k sets B_i, called *blocks*, each of size n, such that each line l of $\mathcal{L}$ meets each block B_i in exactly one point and every pair of points from different blocks lies within exactly one line.

A transversal design with $n = 1$ contains a single line. A transversal design with $k = 1$ is an empty graph, and a transversal design with $k = 2$ is a complete bipartite graph. No more need be said about these cases.

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A transversal design with $k = 3$ is a *Latin square design* (also *dual* 3-*net* or *dual lattice design*). To see the connection with Latin squares, label the three blocks R (for "row"), C (for "column"), and E (for "entry"). Then we can construct an $n \times n$ Latin square by placing the entry e in the cell at row r and column c precisely when $\{r, c, e\} \in \mathcal{L}$, with $r \in R$, $c \in C$, and $e \in E$. The construction can be reversed, so that every Latin square gives rise to a Latin square design.

If n is not 1, then the automorphism group of a Latin square design $TD(3, n)$ (or indeed any transversal design with $k \geq 2$) is not primitive, since it must respect the noncollinearity relation, whose equivalence classes are the various blocks. If the group is transitive, then it has rank at least 3 on points since the stabilizer of a point leaves invariant the remaining points of that block and the union of the remaining blocks.

We are after

(1.1) THEOREM. *Let $\mathbb{L}$ be a Latin square design $TD(3, n)$. Then* Aut$(\mathbb{L})$ *has rank at most* 3 *on points if and only if $n = p^a$ is a power of a prime and $\mathbb{L}$ is isomorphic to the Thomsen design $\mathbb{T}(A)$ of an elementary abelian p-group A of order p^a.*

Here, for a group G, the Thomsen design $\mathbb{T}(G)$ is the Latin square design with point set $\mathcal{P} = G \times \{R, C, E\}$ and line set $\mathcal{L} = \{\, (x_R, y_C, z_E) \,|\, xyz = 1 \,\}$. Its three blocks are $G_R = G \times R$, $G_C = G \times C$, and $G_E = G \times E$. The associated Latin square is the multiplication table of G. Note that we have inverted the entries by taking $xy = z^{-1}$ instead of $xy = z$. With this convention and an abelian group $(G, +)$, the subscript permutations $Sym(R, C, E)$ give automorphisms of $\mathbb{T}(G)$, since $x + y + z = 0$ if and only if $z + x + y = 0$, and so forth.

Some of the arguments used for the classification of rank 3 Latin square designs go through for rank 3 transversal designs $TD(k, n)$ with arbitrary line size $k\,(\geq 3)$, but the corresponding generalization of Theorem 1.1 would require new ideas as well. All conclusions known to us are related to rank 3 translation planes. It is elementary that $k \leq n + 1$ with equality if and only if we have a dual affine plane. A rank 3 dual affine plane must be Desarguesian [7, 4.3.16]. The next case is that of $TD(n, n)$, which are affine planes with one parallel class distinguished as the set of blocks. Rank 3 affine planes of this sort are translation planes and have been classified but need not be Desarguesian. In particular (see [3]), the likeable Walker plane of order 25 gives an example of a rank 3 transversal design $TD(25, 25)$ with nonsolvable rank 3 automorphism group that admits shears but does not have a shear block. (See Section 2 for the appropriate definitions.)

A related result is

(1.2) THEOREM. (BAILEY [2]) *Let $\mathbb{L}$ be a Latin square design. Then* Aut$^\circ(\mathbb{L})$ *has at most four orbits on ordered pairs of distinct lines if and only if $\mathbb{L}$ is $\mathbb{T}(A)$ where either A is an elementary abelian* 2*-group or A is cyclic of order* 3.

Here $\mathrm{Aut}^\circ(\mathbb{L})$ denotes the normal subgroup of $\mathrm{Aut}\,(\mathbb{L})$ (having index at most 6) that leaves each block invariant. Three of the four orbits correspond to pairs that intersect in one of R, C, or E, and the fourth orbit consists of pairs of disjoint lines. We can extend Bailey's result to

(1.3) THEOREM. *Let $\mathbb{L}$ be a Latin square design. Then $\mathrm{Aut}\,(\mathbb{L})$ has one orbit on ordered pairs of distinct intersecting lines or $\mathrm{Aut}^\circ(\mathbb{L})$ has three orbits on ordered pairs of distinct intersecting lines if and only if $\mathbb{L}$ is $\mathbb{T}(A)$ for A an elementary abelian p-group for some prime p.*

Bailey's proof of Theorem 1.2 is elementary, whereas our proofs of Theorems 1.1 and 1.3 use the classification of finite doubly transitive groups (see [4]). Bailey in fact only assumes that $\mathrm{Aut}^\circ(\mathbb{L})$ has at most four orbits on unordered pairs of distinct intersecting lines. We conjecture that a Latin square design whose automorphism group is transitive on each of the three sets

$$\Lambda_B = \{\,\{l_1, l_2\}\,|\, l_i \in \mathcal{L},\, l_1 \cap l_2 = b \in B\,\}\,,$$

for $B \in \{R, C, E\}$, must be $\mathbb{T}(A)$ for some elementary abelian p-group A. This includes Bailey's original theorem and its obvious extension as in Theorem 1.3. It additionally covers Latin square designs admitting automorphism groups that are rank 3 on lines. The methods of this paper should extend to this situation, and we hope to return to its study at some later date.

If $G \leq Sym(\Omega)$ and $\Delta \subseteq \Omega$, then G_Δ is the global stabilizer of Δ in G, $G_{[\Delta]}$ is the pointwise stabilizer in G of Δ, and $G^\Delta = G_\Delta/G_{[\Delta]}$ is the group induced by G on Δ. If Σ is a set of blocks of imprimitivity for G on Ω, then $G_{[\Sigma]}$ is the normal subgroup of G stabilizing all blocks setwise and $G^\Sigma = G/G_{[\Sigma]}$ is the group induced by G on Σ. In the Latin square design case above, $G_{[\Sigma]} = \mathrm{Aut}^\circ(\mathbb{L})$.

2 Shears and a characterization of the Thomsen design of a group

(2.1) LEMMA. *Let $\mathbb{T} = (\mathcal{P}, \mathcal{L})$ be a transversal design $TD(k, n)$ (with $k \geq 3$) with block set Σ. For $G \leq \mathrm{Aut}\,(\mathbb{T})$ and block $B \in \Sigma$, the group $G_{[\Sigma]} \cap G_{[B]}$ is semiregular on each block $D \neq B$.*

PROOF. If $g \in G_{[\Sigma]} \cap G_{[B]}$ fixes the point $d \in D$, then g fixes all lines l through d; so, as $g \in G_{[\Sigma]}$, it fixes all the points $l \cap B'$ for $B' \in \Sigma$ with $B \neq B' \neq D$. Therefore g fixes all points except possibly those of D. Replacing D by B', we see that g also fixes D pointwise; so $g = 1$ as claimed.

Note that this is false if $k = 2$.

In the situation of the lemma, an element of the group $G_{[\Sigma]} \cap G_{[B]}$ is called a *shear* with *axis* B. The lemma says that the group of shears with axis B is semiregular on each block other than B. We say that B is a *shear block* if its group of shears is in fact regular on some other block and so on all other blocks.

The following is equivalent to a result of Praeger [8].

(2.2) THEOREM. *Let $\mathbb{L}$ be a Latin square design $TD(3,n)$ possessing a shear block B with associated group of shears A. Then $\mathbb{L}$ is isomorphic to the Thomsen design $\mathbb{T}(A)$.*

PROOF. We will define an isomorphism i from $\mathbb{L}$ onto the Thomsen design $\mathbb{T}(A)$. The design $\mathbb{L}$ has a shear block B. Let us call the two other blocks F and D. First choose a line l of $\mathbb{L}$, and let $f = l \cap F$, $d = l \cap D$. If u is a point of $\mathbb{L}$ in F, we know, since A acts regularly on F, that there exists exactly one element $x \in A$ mapping f onto u. Define $u^i = x_R$. Similarly, if $v \in D$, there exists exactly one element $y \in A$ mapping v onto d. Define $v^i = y_C$. Now consider $w \in B$. The line on w and d intersects F in t, say. If $t^i = z_R$, then we define $w^i = (z^{-1})_E$. It is easy to see that i is a bijection from the point set of $\mathbb{L}$ to the point set of $\mathbb{T}(A)$. The only thing we need to check is that i maps any line of $\mathbb{L}$ onto a line of $\mathbb{T}(A)$. So let us keep the notation above and assume $\{u, v, w\}$ is a line of $\mathbb{L}$. The image of this line under i is $\{x_R, y_C, (z^{-1})_E\}$. In order for this set to be a line of $\mathbb{T}(A)$, we need $(xy)z^{-1} = 1$. The shear y maps v onto d while fixing w, hence it maps the line $\{u, v, w\}$ onto the line $\{t, d, w\}$ and consequently u onto t. Thus $f^{xy} = u^y = t$, and the shear xy takes f to t. On the other hand, z also is a shear of A mapping f onto t. Since A acts regularly on F we must have $xy = z$, which is the equality we wanted.

We next verify some homogeneity properties for Thomsen designs of groups, particularly those of elementary abelian groups as discussed in Theorems 1.1 through 1.3

(2.3) LEMMA. *Let G be a group and set $\mathbb{L} = \mathbb{T}(G)$. Then the following bijections are in $Aut(\mathbb{L})$. Indeed the automorphims α_g and σ_h are in $Aut^\circ(\mathbb{L})$.*

(1) $\alpha_g \colon x_B \longrightarrow \begin{cases} x_R & \textit{if } B = R \\ (xg)_C & \textit{if } B = C \\ (g^{-1}x)_E & \textit{if } B = E \end{cases}$, *where* $g \in G$.

(2) $\sigma_h \colon x_B \longrightarrow (x^h)_B$ *for* $B \in \{R, C, E\}$, *where* $h \in Aut(G)$.

(3) $\tau \colon x_B \longrightarrow \begin{cases} x_E & \textit{if } B = R \\ x_R & \textit{if } B = C \\ x_C & \textit{if } B = E \end{cases}$.

(4) $\rho \colon x_B \longrightarrow \begin{cases} (x^{-1})_R & \textit{if } B = R \\ (x^{-1})_E & \textit{if } B = C \\ (x^{-1})_C & \textit{if } B = E \end{cases}$.

(2.4) PROPOSITION. *Let $\mathbb{L} = \mathbb{T}(G)$ be the Thomsen design of the group $G \neq 1$.*

(1) $\mathrm{Aut}\,(\mathbb{L})$ *is transitive on the points of $\mathbb{L}$, and the stabilizer of a point in the block B is transitive on the points not in B. The subgroup* $\mathrm{Aut}^\circ(\mathbb{L})$ *is transitive on lines and on each block.*

(2) *If additionally G is an elementary abelian p-group, then* $\mathrm{Aut}\,(\mathbb{L})$ *is rank 3 on the points of $\mathbb{L}$ and has one orbit on ordered pairs of distinct intersecting lines. Its subgroup* $\mathrm{Aut}^\circ(\mathbb{L})$ *has three orbits on ordered pairs of distinct intersecting lines.*

(3) *If additionally the elementary abelian p-group is either a* 2*-group or has order* 3*, then* $\mathrm{Aut}^\circ(\mathbb{L})$ *is transitive on ordered pairs of disjoint lines.*

Proof. (1) Let A be the subgroup of $H = \mathrm{Aut}^\circ(\mathbb{L})$ generated by all the α_g, which is transitive on C and E. Then the subgroup generated by A and τ is transitive on the points of $\mathbb{L}$, while the subgroup generated by A and ρ stabilizes Id_R and is transitive on $C \cup E$. The group $H \geq A^\tau$ is transitive on R, and $H \geq A$ can map the line $\{Id_R, Id_C, Id_E\}$ onto any line through Id_R.

(2) Suppose in addition that G is a nontrivial elementary abelian p-group, which we identify with the additive group of the field $\mathbb{F}_q$. For $0 \neq a \in G$, let $\mu(a)$ be the automorphism of G given by $x \mapsto a.x$. Using $\sigma_{\mu(a)}$ for all choices of a, we see that the stabilizer in $\mathrm{Aut}^\circ(\mathbb{L})$ of 0_R is transitive on $R \setminus \{0_R\}$. Therefore $\mathrm{Aut}\,(\mathbb{L})$ is rank 3 on the points of $\mathbb{L}$.

Since τ normalizes line-transitive H, it remains to prove that the stabilizer in H of $L = \{0_R, 0_C, 0_E\}$ is transitive on all other lines through 0_R. The automorphism $\sigma_{\mu(a)}$ stabilizes L and maps the line $\{0_R, 1_C, -1_E\}$ onto $\{0_R, a_C, -a_E\}$. As any line through 0_R is of this form, this proves (2).

(3) By (1) the group H is transitive on lines, so we need only prove that the stabilizer in H of $L = \{0_R, 0_C, 0_E\}$ is transitive on all lines disjoint from L.

When $q = 3$, there are only two lines disjoint from L, namely $\{1_R, 1_C, 1_E\}$ and $\{2_R, 2_C, 2_E\}$. These are switched by $\sigma_{\mu(2)}$, which belongs to H and also stabilizes L.

When $q = 2^n$, $\mathrm{Aut}\,(G)$ contains $GL(n,2)$, which is 2-transitive on $G \setminus \{0\}$. Let $\{a_R, b_C, (a+b)_E\}$ and $\{c_R, d_C, (c+d)_E\}$ be two lines disjoint from L. (This means a, b, c, and d are nonzero with $a \neq b$ and $c \neq d$.) Then there exists an automorphism h of G stabilizing 0 (of course), mapping a onto c and b onto d. Then $\sigma_h \in H$ maps the first line onto the second and fixes L, giving the desired conclusion.

3 Basics

The first two results are elementary.

(3.1) Proposition. *Let G be rank* 3 *and imprimitive on* Ω *with block set* Λ*. Then* G^Λ *is* 2*-transitive on* Λ *and* G^B *is* 2*-transitive on each block* B *of* Λ*.*

(3.2) Lemma. *Let* $G \leq \mathrm{Aut}\,((\mathcal{P}, \mathcal{L}))$ *be rank* 3 *on the points of the proper partial linear space* $(\mathcal{P}, \mathcal{L})$*. Then G is transitive on ordered pairs of collinear points and, in particular, is transitive on the line set* $\mathcal{L}$ *and* 2*-transitive on each line.*

Therefore the groups we will be examining are pasted together from 2-transitive groups. The following result of Burnside provides the basic case division. (Recall that the socle of a group G, written $Soc(G)$, is the product of all its minimal normal subgroups.)

(3.3) THEOREM. (BURNSIDE [4, THEOREM 4.3].) *If finite $G \leq Sym(\Omega)$ is 2-transitive on the set Ω, then the socle of G is its unique minimal normal subgroup and we have one of*

(1) *$Soc(G)$ is nonabelian simple and is primitive on Ω;*

(2) *$Soc(G)$ is an elementary abelian p-group, for some prime p, and is regular on Ω.*

In the first case G is *almost simple* and in the second case G is *affine*, indeed, *p-affine*.

The results we use from the classification of finite 2-transitive groups are collected in

(3.4) PROPOSITION. *Let finite $G \leq Sym(\Omega)$ be 2-transitive on the set Ω with $|\Omega| > 2$.*

(1) *Assume that we are in the almost simple case. Then G has a unique minimal normal 2-transitive subgroup N. This N has at most two nonisomorphic representations as a 2-transitive group of this degree.*

The group N is equal to the simple socle $Soc(G)$ except when $G = N$ is $P\Gamma L_2(8)$ acting 2-transitively on 28 points and $Soc(G)$ equals $PSL_2(8)$, which has index 3 and is primitive of rank 4. The group $PSL_2(8)$ has only one isomorphism class of faithful permutation representations of degree 28.

(2) *Assume that we are in the p-affine case. Let M have index at most 2 in G.*

If M contains more than two conjugacy classes of complements to the regular normal subgroup $O_p(G) = O_p(M)$, then, for each point b of Ω, the subgroup $E = Soc(G_b) = Soc(M_b)$ is a simple group $Sp_{2m}(q)$ or $G_2(q)$ with even $q(\geq 4)$. In this case, E has orbits of different lengths on each class of complements to $O_p(M)$ in $O_p(M)E$.

PROOF. (1) Almost all of this can be found in [4, Table 7.4]. A degree 28 permutation representation of $PSL_2(8)$ has as point stablizer the normalizer of a Sylow 3-subgroup and so is uniquely determined.

(2) Assume that M has more than two classes of complements to $V = O_p(M) = O_p(G)$. If M has index 2 in G, then G still has more than one class of complements. From [4, Table 7.3], we learn that $E = Soc(M_b) = Soc(G_b)$ is $Sp_{2m}(q)$ or $G_2(q)$ with $4 \leq q$ even and that V is $\mathbb{F}_q^{2m}$ or $\mathbb{F}_q^6$, respectively. Here M_b is a fixed complement to V in M. Set $m = 3$ when E is $G_2(q)$. For (2) it remains to calculate orbit lengths for E on each class of complements.

The subgroup VE is itself 2-transitive on Ω and contains exactly q conjugacy classes of complements by [6]. By standard cohomological results, there is a uniquely determined $\mathbb{F}_qE$-module $W \geq V = [W, E]$ with $\dim_{\mathbb{F}_q}(W/V) = \dim_{\mathbb{F}_q}(H^1(E, V)) = 1$ and such that the semidirect product WE has a unique conjugacy class of complements to W. These q^{2m+1} complements are the E^w, for $w \in W$. The VE-class containing E^w is $(E^w)^V = (E^V)^w$ of size q^{2m}, and so the action of E on a class of complements is isomorphic to its action on the corresponding coset $V + w$ of V. If the coset is V itself then the orbit lengths are 1 and $q^{2m}-1$, so we are done in this case and now need only treat cosets $V+w \neq V$.

First consider E of type $Sp_{2m}(q)$. Let Q be a nondegenerate quadratic form defined on a $\mathbb{F}_q$-space W_0 of dimension $2m+2$. In the full isometry group $O(W_0,Q)$, the stabilizer of the nonsingular 1-space X is $R \times O_{2m+1}(q)$, where R is the reflection subgroup of order 2 with center X. As q is a power of 2 at least 4, the perfect factor $O_{2m+1}(q)$ is isomorphic to $Sp_{2m}(q)$, acts trivially on X, and can be identified with E. The $\mathbb{F}_qE$-module W can then be taken to be W_0/X with $V = X^{\perp}/X$.

Let $V+w\,(\neq V)$ be a coset of V in W. Then the stabilizer of w in E is equal to the centralizer in $O(W_0,Q)$ of the nondegenerate 2-space U of W_0 spanned by X and w_0 (a preimage of w). This subgroup of E is $O^{\epsilon}_{2m}(q)$, of Witt type $\epsilon = +$ or $-$ depending upon whether $U^{\perp}$ is a hyperbolic or elliptic hyperplane of $X^{\perp}$. Hyperplanes of both types exist in $X^{\perp}$, so it is possible to choose w_0, representing a member of $V+w$, to realize either type as $U^{\perp}$. That is, in the coset $V+w$ it is possible to find representatives with E-orbit length $|Sp_{2m}(q)\colon O^{+}_{2m}(q)|$ and representatives with E-orbit length $|Sp_{2m}(q)\colon O^{-}_{2m}(q)|$. These numbers are different, completing (2) in the symplectic case.

The argument for $G_2(q)$ with q a power of 2 (at least 4) is similar. Let W_0 be an 8-dimensional split Cayley algebra over $\mathbb{F}_q$ admitting composition with respect to the nondegenerate hyperbolic quadratic form Q. We can again take W to be W_0/X with $V = X^{\perp}/X$, where X is the nonsingular 1-space spanned by the identity element of the algebra W_0. The automorphism group $E \simeq G_2(q)$ of the algebra W_0 is then naturally a subgroup of the corresponding subgroup $Sp_6(q)$ of the previous two paragraphs. Again the stabilizers in E of hyperbolic and elliptic hyperplanes of $X^{\perp}$ have different orders (see [1, Theorems 1 and 3]) and so give rise to E-orbits of different length in cosets of V in W.

4 Rank 3 Latin square designs

Let $\mathbb{L} = (\mathcal{P},\mathcal{L})$ be a Latin square design with block set $\Sigma = \{R,C,E\}$. Set $G = \mathrm{Aut}\,(\mathbb{L})$ and $M = G_{[\Sigma]} = \mathrm{Aut}^{\circ}(\mathbb{L})$.

Let $S_B = G_{[\Sigma]} \cap G_{[B]}$, the group of shears with axis $B \in \Sigma$, and let S be the subgroup generated by all the S_B for $B \in \Sigma$.

Consider two hypotheses:

(4.1) HYPOTHESIS. *G is rank 3 on the points of $\mathcal{P}$.*

and

(4.2) HYPOTHESIS. *M has exactly three orbits on ordered pairs of intersecting distinct lines from $\mathcal{L}$.*

We then have easily

(4.3) PROPOSITION.

(1) *Under Hypothesis 4.1:*

(*i*) *For all blocks B, G^B is 2-transitive on B and contains M^B with index at most 2;*

(*ii*) *For each point b and block B with $b \notin B$, the stabilizer M_b is transitive on B.*

(2) *Under Hypothesis 4.2:*

(*i*) *For all blocks B, M^B is 2-transitive on B;*

(*ii*) *For each point b and block B with $b \notin B$, the stabilizer M_b is transitive on B.*

In view of the similarities in the proposition, we treat the two hypotheses simultaneously. Theorems 1.1 and 1.3 follow immediately from Proposition 2.4 and

(4.4) Theorem. *Let $\mathbb{L}$ be a proper Latin square design satisfying one of the Hypotheses 4.1 or 4.2. Then $\mathbb{L}$ is $\mathbb{T}(A)$ for some elementary abelian p-group A.*

(4.5) Lemma. *A Latin square design $TD(3,n)$ with $n \in \{1,2,3\}$ is $\mathbb{T}(A)$ for A cyclic of order n.*

Proof. This result is trivial for $n = 1$ and well-known for $n = 2, 3$. Indeed any $TD(3,2)$ is a dual affine plane of order 2, and any $TD(3,3)$ is an affine plane of order 3 with one parallel class identified as blocks.

We may therefore assume that $\mathbb{L}$ is a $TD(3,n)$ with $n \geq 4$.

(4.6) Lemma. *For each block B, the group M^B is primitive on B.*

Proof. Under (4.2) this is immediate. Under (4.1) this also holds as M^B has index at most 2 in 2-transitive $G^B \leq Sym(B)$.

(4.7) Lemma. *For some block B, the subgroup S_B is nontrivial.*

Proof. Assume, for a contradiction, that $S = 1$, so that M is faithful on each block B.

First suppose that G^B is almost simple. Therefore by Proposition 3.4(1) $Soc(M)(\simeq Soc(G^B))$ is simple, and $Soc(M)$ is either 2-transitive on each $D \in \Sigma$ or $Soc(M) \simeq PSL_2(8)$ is primitive on the 28 points of each D. Again by Proposition 3.4(1), there are at most two possibilities, up to isomorphism, for the permutation groups $(Soc(M), B)$ with $B \in \Sigma$. In particular, since $|\Sigma| = 3$, there are at least two blocks, B and D (say), on which $Soc(M)$ has isomorphic representations. For a point $b \in B$, the point stablizer $Soc(M)_b$ also then fixes a unique point $d \in D$ and so a unique line, namely the line l on b, d. However M_b normalizes $Soc(M)_b$, which fixes a unique line l on b; so M_b also fixes l. But this is a contradiction, since M_b is transitive on the lines through b.

The proof in the affine case is similar to the almost simple case. By primitivity (Lemma 4.6), for a point $b \in B$, the only point of B that is stabilized by M_b is b itself.

Arguing as before, we are done if two of the representations of M on the various $B \in \Sigma$ are isomorphic. Therefore by Proposition 3.4(2) we are done unless, for each $b \in B$, the subgroup $Soc(M_b)$ is $Sp_{2m}(q)$ or $G_2(q)$ with q even and at least 4. Again, the stabilizer M_b is transitive on the lines through b and normalizes $Soc(M_b)$, so the orbits of $Soc(M_b)$ on the lines through b must have uniform length. This in turn implies that the orbits of $Soc(M_b)$ on each block not containing b also have uniform length. But that is not the case by Proposition 3.4(2).

Proof of Theorem 4.4:

For some block B we have $S_B \neq 1$ by Lemma 4.7. Let D be a block other than B. By Lemma 2.1, $S_B \cap M_{[D]} = 1$ and $S_B = S_B / S_B \cap M_{[D]} \simeq S_B^D$ is semiregular, nontrivial, and normal in M^D. As M^D is primitive by Lemma 4.6, S_B^D is actually regular on D. That is, B is a shear block for $\mathbb{L}$. By Theorem 2.2, $\mathbb{L}$ is isomorphic to $\mathbb{T}(S_B)$. Here $S_B^D \simeq S_B$ is regular and subnormal in 2-transitive G^D or M^D and so is an elementary abelian p-group by Theorem 3.3. This gives the theorem.

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