Adv. Geom. **6** (2006), 225–241

Advances in Geometry

Extensions of isomorphisms for affine Grassmannians over $\mathbb{F}_2$

Rieuwert J. Blok* and Jonathan I. Hall

(Communicated by A. Pasini)

Abstract. In Blok [1] affinely rigid classes of geometries were studied. These are classes B of geometries with the following property: Given any two geometries $\Gamma_1, \Gamma_2 \in \mathsf{B}$ with subspaces $\mathscr{S}_1$ and $\mathscr{S}_2$ respectively, then any isomorphism $\Gamma_1 - \mathscr{S}_1 \to \Gamma_2 - \mathscr{S}_2$ uniquely extends to an isomorphism $\Gamma_1 \to \Gamma_2$.

Suppose Γ belongs to an affinely rigid class. Then for any subspace $\mathscr{S}$ we have $\mathrm{Aut}(\Gamma - \mathscr{S}) \leqslant \mathrm{Aut}(\Gamma)$. Suppose that, in addition, Γ is embedded into the projective space $\mathbb{P}(V)$ for some vector space V. Then one may think of V as a "natural" embedding if every automorphism of Γ is induced by some (semi-) linear automorphism of V. This is for instance true of the projective geometry $\Gamma = \mathbb{P}(V)$ itself by the fundamental theorem of projective geometry. Clearly since Γ belongs to an affinely rigid class and has a natural embedding into $\mathbb{P}(V)$, also the embedding $\Gamma - \mathscr{S}$ into $\mathbb{P}(V)$ is natural.

In Blok [1] the notion of a layer-extendable class was introduced and it was shown that layer-extendable classes are affinely rigid. As an application, it was shown that the union of most projective geometries, (dual) polar spaces, and strong parapolar spaces forms an affinely rigid class. However, the geometries motivating that study, the Grassmannians defined over $\mathbb{F}_2$, were not included in this class because they do not form a layer-extendable class. Since affine projective geometries (1-Grassmannians, if you will) are simply complete graphs, clearly they are not affinely rigid at all. In the present note we show that also the class of 2-Grassmannians over $\mathbb{F}_2$ fails to form an affinely rigid class, although in a less dramatic way, whereas the class of k-Grassmannians of projective spaces of dimension n over $\mathbb{F}_2$ where $3 \leqslant k \leqslant n-2$ is in fact affinely rigid.

Key words. Grassmannian, parapolar spaces, hyperplane complements, affine geometry, isomorphism extensions, Fundamental Theorem of Projective Geometry.

2000 Mathematics Subject Classification. 51A45, 14R10, 51A15, 51A10

1 Introduction

A *point-line geometry* is a pair $\Gamma = (\mathscr{P}, \mathscr{L})$, where $\mathscr{P}$ is a set whose elements are called *points* and $\mathscr{L}$ is a set whose elements are subsets of $\mathscr{P}$ called *lines*. A point-

*Supported in part by grant HPRN-CT-2001-00272.

line geometry Γ is a *partial linear space*, if any two points are contained in at most one line. We call Γ *thick*, if every line has at least three points. Note that this means that a grid, although not thick as a building, is thick as a point-line geometry in the sense defined here. Throughout the paper we will assume that point-line geometries are partial linear and thick, unless specified otherwise.

Given a point-line geometry $\Gamma = (\mathscr{P}, \mathscr{L})$, let X be any subset of $\mathscr{P}$. If $|L \cap X| \geqslant 2$ for some line $L \in \mathscr{L}$, then we call this intersection a *line of* X. The collection of all lines of X is denoted $\mathscr{L}(X)$. We call X a *subspace* if all lines of X are in fact lines of Γ. The subspace X is *proper* if $\emptyset \neq X \neq \mathscr{P}$. A (geometric) *hyperplane* of Γ is a proper subspace H with the property that $L \cap H \neq \emptyset$ for all $L \in \mathscr{L}$. Hyperplanes are "large" and are often, but not always, maximal subspaces with respect to containment.

Given a subspace $\mathscr{S}$ of Γ, by $\Gamma - \mathscr{S}$ we denote the point-line geometry induced by Γ on the point-set $\mathscr{P} - \mathscr{S}$.

We recall the following definition from Blok [1].

Definition 1.1. A class B of point-line geometries is called *affinely rigid* (AR) if and only if

(AR) given $\Gamma_i \in \mathsf{B}$ with a subspace $\mathscr{S}_i$ $(i = 1, 2)$, then any isomorphism $\Gamma_1 - \mathscr{S}_1 \to \Gamma_2 - \mathscr{S}_2$ extends uniquely to an isomorphism $\Gamma_1 \to \Gamma_2$.

Grassmannians are mostly affinely rigid. We will now discuss the geometries under study in this note and state the main results. Let Δ be the building of type A_n over the field $\mathbb{F}$. This is the incidence geometry whose *objects of type i* (for all $1 \leqslant i \leqslant n$) are the *i*-spaces of some vector space V of dimension $n+1$ over $\mathbb{F}$ and in which two objects are incident whenever one contains the other as a subspace. Recall that a *flag* F is a set of pairwise incident elements and that $\mathrm{typ}(F)$ is the set of types of elements occurring in F.

The k-shadow space of Δ is also called an (n,k)-Grassmannian over $\mathbb{F}$, or simply an $A_{n,k}(\mathbb{F})$ geometry. It is is the point-line geometry $\Gamma = (\mathscr{P}, \mathscr{L})$ whose points are the k-spaces of V and whose lines are pairs (B, U), where B is a $(k-1)$-space and U is a $(k+1)$-space of V such that $B \leqslant U$, and in which a point P belongs to a line (B, U) if and only if $B \leqslant P \leqslant U$.

The (n,k)-Grassmannians over $\mathbb{F}_2$ are the main object of this study. We ask which families of Grassmannians over $\mathbb{F}_2$ form an affinely rigid class. Many other geometries, including the Grassmannians defined over any field other than $\mathbb{F}_2$, were already considered in Blok [1]. The Grassmannians defined over $\mathbb{F}_2$ however form the main case missing from that paper.

In order to phrase the answer it is convenient to distinguish the following subfamilies of Grassmannians over $\mathbb{F}$. For any $l \in \mathbb{N}_{>0}$, let $\mathsf{A}_l(\mathbb{F})$ denote the class of all $A_{n,k}$ geometries such that $k = l$ or $k = n+1-l$. Thus $\mathsf{A}_1(\mathbb{F})$ is the class of (dual) projective spaces and $\mathsf{A}_2(\mathbb{F})$ is the class of projective (dual) line-Grassmannians. Also, for $m \in \mathbb{N}_{>0}$, let $\mathsf{A}_{\geqslant m}(\mathbb{F}) = \bigcup \mathsf{A}_{n,k}(\mathbb{F})$ where the union runs over all n and k with $m \leqslant k \leqslant n+1-m$. For a finite prime power q we abbreviate $\mathsf{A}_l(\mathbb{F}_q)$ by $\mathsf{A}_l(q)$, and so on.

Clearly the class $\mathsf{A}_1(2)$ is not affinely rigid. Given a projective space Γ of projective dimension n and hyperplane $\mathscr{S}$, the geometry $\Gamma - \mathscr{S}$ is just a complete graph on 2^n points so that $\mathrm{Aut}(\Gamma - \mathscr{S}) = \mathrm{Sym}(2^n)$. On the other hand, $\mathrm{Aut}(\Gamma) = \mathrm{SL}_{n+1}(\mathbb{F}_2)$ and $\mathrm{Stab}_{\mathrm{Aut}(\Gamma)}(\Gamma - \mathscr{S}) = 2^n.\mathrm{SL}_n(\mathbb{F}_2)$. For $n \geqslant 3$ the former group is larger than the latter so there are many automorphisms of $\Gamma - \mathscr{S}$ that cannot be extended to an isomorphism of Γ.

A more subtle case is the following.

Theorem 1. *The class* $\mathsf{A}_2(2)$ *of* (*dual*) *line-Grassmannians over* $\mathbb{F}_2$ *is not affinely rigid.*

It turns out that here the gap between $\mathrm{Aut}(\Gamma - \mathscr{S})$ and $\mathrm{Stab}_{\mathrm{Aut}(\Gamma)}(\Gamma - \mathscr{S})$ depends on $\mathscr{S}$ and is generally not very large, and is 0 whenever $\mathscr{S}$ is an attenuated hyperplane. The following result settles the affine rigidity for all remaining Grassmannians over $\mathbb{F}_2$.

Theorem 2. *The class* $\mathsf{A}_{\geqslant 3}(2)$ *of all* (n,k)*-Grassmannians over* $\mathbb{F}_2$ *such that* $3 \leqslant k \leqslant n-2$ *is affinely rigid.*

In Section 2 we prove Theorem 2.5 which provides a method for showing that a class of geometries that contains many *LE*-subgeometries is itself *LE* (see Definition 2.2). This is a generalization of Theorem 4.3 of Blok [1].

In Section 3 we study some general properties of a Grassmannian Γ that are uniquely determined by $\Gamma - \mathscr{S}$. For instance, Lemma 3.4 shows that given an A_{n_i,k_i}-geometry Γ_i, $i = 1, 2$, with subspace $\mathscr{S}_i$ and an isomorphism $\varphi : \Gamma_1 - \mathscr{S}_1 \to \Gamma_2 - \mathscr{S}_2$, it follows that $\Gamma_1, \Gamma_2 \in \mathsf{A}_{n,k}$ where $n = n_1 = n_2$ and $k_1, k_2 \in \{k, n+1-k\}$.

In Section 4 we prove Theorem 1 by explicitly calculating the index $[\mathrm{Aut}(\Gamma - \mathscr{S}) : \mathrm{Stab}_{\mathrm{Aut}(\Gamma)}(\Gamma - \mathscr{S})]$ in the case that $\mathscr{S}$ is a hyperplane. This index is governed by the size of the radical of the symplectic form defining the hyperplane. Also, our Theorem 4.6 answers the following question of Shult [7] in the affirmative.

Question 1.2. Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be hyperplanes of the (n,k)-Grassmannian Γ with underlying vector space V. If the affine Grassmannians $\Gamma - \mathscr{S}_1$ and $\Gamma - \mathscr{S}_2$ are isomorphic, does there exist an element of $\mathrm{P\Gamma L}(V)$ that induces an isomorphism of $\Gamma - \mathscr{S}_1$ and $\Gamma - \mathscr{S}_2$?

In Section 5 we prove Theorem 2 using Theorem 2.5.

Further notation. The objects of Δ of type $k-1$, k, $k+1$ will be referred to as objects of type $-$, 0, $+$, respectively; to objects of type $k+i$ and $k-i$ with $i \geqslant 2$ we refer as objects of type $+i$ and $-i$.

We will want to have the following notation available to us. However, to avoid overly cumbersome notation, we will only use it to avoid possible confusion. For any flag F of Δ and $\tau \in \{-k, \dots, -, 0, +, \dots, n-k\}$, let $[F]_\tau$ denote the τ-shadow of F, that is, the set of objects of type τ incident with F in Δ. Since in $\Gamma - \mathscr{S}$ a flag F of Δ is only represented by its set of points off $\mathscr{S}$ we need special notation for shadows of other types in $\Gamma - \mathscr{S}$. More precisely, for τ as above, let $[F]_{\tau a}$ be the set of objects of type τ that are incident with F and some point of $\Gamma - \mathscr{S}$ (a for "affine").

Example 1.3. If L is a line of Γ, then in fact L is a flag $([L]_-, [L]_+)$ of type $(-, +)$. Also, $[L]_0$ is the set of points of Γ on L and $[L]_{0,a}$ is the set of points of $\Gamma - \mathscr{S}$ on L.

The *collinearity graph* of a point-line geometry $\Gamma = (\mathscr{P}, \mathscr{L})$ is the graph with vertex set $\mathscr{P}$ and in which two vertices are adjacent if and only if the corresponding points are collinear. We call a point-line geometry *connected* if its collinearity graph is connected. The *distance* $\mathrm{d}(X, Y)$ between points X and Y is the length of a shortest path from X to Y in the collinearity graph of Γ. The *diameter* is the integer diam $= \max\{\mathrm{d}(X, Y) \mid X, Y \in \mathscr{P}\}$ if it is finite, and diam $= \infty$ otherwise. We say that two lines L and M are *concurrent* if they intersect in a point; we write $L * M$.

A *singular subspace* of a point-line geometry is a subspace any two points of which are collinear. A set of points C is called *convex* if any geodesic in the collinearity graph between two points of the subspace is entirely contained in the collinearity graph of that subspace. The *convex closure* of a set of points X is the smallest convex subspace containing X.

A *symplecton* is a subspace isomorphic to a non-degenerate polar space of rank at least 2 that is the convex closure of any two of its points at mutual distance 2.

2 Layer-extendable classes

Given Γ with subspace $\mathscr{S}$, a point $P \in \mathscr{S}$ is *non-deep* in $\mathscr{S}$ if it is collinear to some point of $\Gamma - \mathscr{S}$. We denote the set of non-deep points in $\mathscr{S}$ by $\mathscr{N}(\mathscr{S})$; this set is sometimes called the *boundary* of S. The elements of $\mathscr{D}(\mathscr{S}) = \mathscr{S} - \mathscr{N}(\mathscr{S})$ are called *deep* in $\mathscr{S}$. The following refinement of this notion will be crucial in this paper.

Definition 2.1. Following Shult [6] we define a sequence of subsets $D_i(\mathscr{S})$ as follows: Set $D_{-1}(\mathscr{S}) = \mathscr{P} - \mathscr{S}$, let $D_0(\mathscr{S})$ be the set of non-deep points of $\mathscr{S}$, and for $i \geqslant 0$ define

$$D_{i+1}(\mathscr{S}) = \{P \in \mathscr{P} \mid P \text{ is collinear to a point of } D_i(\mathscr{S}) \text{ but not to any point of } D_{i-1}(\mathscr{S})\}.$$

We then set

$$D_i^*(\mathscr{S}) = \bigcup_{j=i}^{\infty} D_j(\mathscr{S}).$$

Given a line L of Γ and some subset of points $l \subseteq L$, if $|l| \geqslant 2$ then, since Γ is a partial linear space, l determines the line L uniquely. We say that l *supports* L and write $\bar{l} = L$. We call a line L of $\Gamma - \mathscr{S}$ *short* if $\bar{L} \neq L$. Note that in this case $\bar{L} \cap \mathscr{S}$ consists of exactly one point. Lines of $\Gamma - \mathscr{S}$ that are not short are called *long*.

(AutoPDF V7 11/1/06 10:41) WDG (170×240mm) TMaths J-1445 Adv. in Geom. 6:2 PMU:I(CKN[V])10/1/2006 pp. 225–241 1445_6-2_04 (p. 228)

We recall from Blok [1] the definition of a layer-extendable class of geometries.

Definition 2.2. A *layer-extendable* or *LE*-class is a class B of point-line geometries, which is closed under isomorphisms, with the following properties:

(LE1) Every element of B is a connected thick partial linear space,

(LE2) for every $\Gamma \in \mathsf{B}$ with subspace $\mathscr{S} \subseteq \Gamma$ the set $D_i^*(\mathscr{S})$ is a subspace of Γ for every $i \in \mathbb{N}$,

(LE3) given $\Gamma_i \in \mathsf{B}$ with subspace $\mathscr{S}_i$ $(i = 1, 2)$ and some isomorphism $\varepsilon : \Gamma_1 - \mathscr{S}_1 \to \Gamma_2 - \mathscr{S}_2$, for any two non-intersecting lines $L_1, L_2 \in \Gamma_1 - \mathscr{S}_1$ we have
(LE3.1) $L_1 \neq \overline{L_1}$ if and only if $L_1^\varepsilon \neq \overline{L_1^\varepsilon}$,
(LE3.2) $\overline{L_1} * \overline{L_2}$ if and only if $\overline{L_1^\varepsilon} * \overline{L_2^\varepsilon}$, and
(LE3.3) for any line H_1 with $|H_1 \cap \mathscr{N}(\mathscr{S}_1)| \geqslant 2$ there is a line H_2 with $|H_2 \cap \mathscr{N}(\mathscr{S}_2)| \geqslant 2$ such that $\overline{L_1} - L_1 \in H_1$ if and only if $\overline{L_1^\varepsilon} - L_1^\varepsilon \in H_2$.

Note 2.3. In Blok [1] we did not specify whether or not we understood an *LE*-class to be closed under isomorphism.

The use of this notion is the following result proved in Blok [1].

Theorem 2.4. *Let* B *be a class of point-line geometries satisfying* (LE1) *and* (LE2). *Then* B *is an LE-class if and only if it is affinely rigid.*

We now present a way to discover new *LE*-classes using old *LE*-classes. This is a modified, but considerably more powerful version of Theorem 3.4 of Blok [1].

Theorem 2.5. *Let* B *be a class of point-line geometries satisfying* (LE1) *and* (LE2) *of Definition* 2.2. *Suppose in addition that for any* $\Gamma \in \mathsf{B}$ *with subspace* $\mathscr{S}$ *there is a collection* $\mathscr{T}(\mathscr{S})$ *of subspaces of* Γ *satisfying the following conditions.*

(L) *Every line containing a non-deep point* P *of* $\mathscr{S}$ *is contained in some element of* $\mathscr{T}(\mathscr{S})$, *and*

(IL) *for any two lines* L_1 *and* L_2 *of* $\Gamma - \mathscr{S}$ *such that* $\overline{L_1}$ *and* $\overline{L_2}$ *intersect in a non-deep point* P *of* $\mathscr{S}$, *there is a* k *and a finite sequence*

$$L_1 = M_0, T_0, M_1, T_1, \ldots, T_k, M_{k+1} = L_2$$

where M_i, M_{i+1} *are lines of* $\Gamma - \mathscr{S}$ *belonging to* $T_i \in \mathscr{T}(\mathscr{S})$ *such that* $\overline{M_i}$ *and* $\overline{M_{i+1}}$ *intersect in* P.

Moreover, given $\Gamma_i \in \mathsf{B}$ *with subspace* $\mathscr{S}_i$ $(i = 1, 2)$ *and some isomorphism* $\varepsilon : \Gamma_1 - \mathscr{S}_1 \to \Gamma_2 - \mathscr{S}_2$,

(T) *for every* $T_1 \in \mathcal{T}(\mathcal{S}_1)$, *there is* $T_2 \in \mathcal{T}(\mathcal{S}_2)$ *such that* $(T_1 - \mathcal{S}_1)^{\varepsilon} = T_2 - \mathcal{S}_2$, *and*

(LE) *the set* $\mathcal{T}(\mathcal{S}_1) \cup \mathcal{T}(\mathcal{S}_2)$ *forms an LE-class.*

Then B *is an LE-class.*

Proof. We only have to show that B satisfies (LE3.1)–(LE3.3). Let $\Gamma_j \in \mathsf{B}$ have subspace $\mathcal{S}_j$ $(j = 1, 2)$ and suppose there is an isomorphism $\varepsilon : \Gamma_1 - \mathcal{S}_1 \to \Gamma_2 - \mathcal{S}_2$. Now let L_1, L_2 be non-intersecting lines of $\Gamma_1 - \mathcal{S}_1$.

(LE3.1): Suppose L_1 or L_1^{ε} is short. Without loss of generality we may assume that L_1 is short. By (L) applied to L_1, there exists $T_1 \in \mathcal{T}(\mathcal{S}_1)$ such that T_1 contains L_1. By (T) there is $T_2 \in \mathcal{T}(\mathcal{S}_2)$ such that $\varepsilon(T_1 - \mathcal{S}_1) = T_2 - \mathcal{S}_2$. By (LE), $\mathcal{T}(\mathcal{S}_1) \cup \mathcal{T}(\mathcal{S}_2)$ is an *LE*-class. Now $\varepsilon : T_1 - \mathcal{S}_1 \to T_2 - \mathcal{S}_2$ is an isomorphism and by (LE3.1) applied to the *LE*-class $\mathcal{T}(\mathcal{S}_1) \cup \mathcal{T}(\mathcal{S}_2)$, $\overline{L_1} \neq L_1$ if and only if $\overline{L_1^{\varepsilon}} \neq L_1^{\varepsilon}$. Hence (LE3.1) is satisfied.

(LE3.2): If $\overline{L_1}$ and $\overline{L_2}$ intersect in a (non-deep) point $P \in \mathcal{S}_1$, then by (IL) there is a k and a finite sequence $L_1 = M_0, T_0, M_1, T_1, \ldots, T_k, M_{k+1} = L_2$ where M_i, M_{i+1} are lines of $\Gamma - \mathcal{S}$ in $T_i \in \mathcal{T}(\mathcal{S})$ such that $\overline{M_i}$ and $\overline{M_{i+1}}$ intersect in P.

Fix $i \in \{1, 2, \ldots, k\}$ and set $T = T_i$, $L = M_i$, $N = M_{i+1}$. We then have the following. Since $T \in \mathcal{T}(\mathcal{S}_1)$, by (T) there is $U \in \mathcal{T}(\mathcal{S}_2)$ such that $\varepsilon(T - \mathcal{S}_1) = U - \mathcal{S}_2$. Now $\varepsilon : T - \mathcal{S}_1 \to U - \mathcal{S}_2$ is an isomorphism and by (LE3.2) applied to the *LE*-class $\mathcal{T}(\mathcal{S}_1) \cup \mathcal{T}(\mathcal{S}_2)$, since $\overline{L} * \overline{N}$, also $\overline{L^{\varepsilon}} * \overline{N^{\varepsilon}}$. This holds for any i and since $\overline{M_i^{\varepsilon}} - M_i^{\varepsilon}$ consists of a single point for all i, we find that $\overline{L_1^{\varepsilon}} * \overline{L_2^{\varepsilon}}$. Now the same argument applied to the isomorphism ε^{-1} shows that also $\overline{L_1^{\varepsilon}} * \overline{L_2^{\varepsilon}}$ implies $\overline{L_1} * \overline{L_2}$. Thus (LE3.2) is satisfied.

(LE3.3): Suppose H_1 is a line of Γ_1 with $|H_1 \cap \mathcal{N}(\mathcal{S}_1)| \geqslant 2$. Since H_1 contains a non-deep point of $\mathcal{S}_1$, by (IL), there is an element $T_1 \in \mathcal{T}(\mathcal{S}_1)$ containing H_1. By (T) there is $T_2 \in \mathcal{T}(\mathcal{S}_2)$ such that $\varepsilon(T_1 - \mathcal{S}_1) = T_2 - \mathcal{S}_2$.

Since H_1 contains a non-deep point of $\mathcal{S}_1$ and $\mathcal{D}(T_1 \cap \mathcal{S}_1)$ is a subspace of T_1, at least two points of H_1 are non-deep points of $T_1 \cap \mathcal{S}_1$ in T_1 and hence are non-deep points of $\mathcal{S}_1$ also. Now $\varepsilon : T_1 - \mathcal{S}_1 \to T_2 - \mathcal{S}_2$ is an isomorphism and so by (LE3.3) applied to the *LE*-class $\mathcal{T}(\mathcal{S}_1) \cup \mathcal{T}(\mathcal{S}_2)$, there is a line H_2 in Γ_2 with $|H_2 \cap \mathcal{N}(\mathcal{S}_2)| \geqslant 2$ such that for all short lines L_1 of $T_1 - \mathcal{S}_1$, $\overline{L_1} - L_1 \in H_1$ if and only if $\overline{L_1^{\varepsilon}} - L_1^{\varepsilon} \in H_2$. Now let $L_2 \neq L_1$ be any other short line of $\Gamma_1 - \mathcal{S}_1$ with $\overline{L_2} - L_2 = \overline{L_1} - L_1$. Then since Γ_1 is a partial linear space, $L_1 \cap L_2 = \emptyset$ and since ε is an isomorphism, also $L_1^{\varepsilon} \cap L_2^{\varepsilon} = \emptyset$. By (LE3.2) since $\overline{L_1} * \overline{L_2}$, also $\overline{L_1^{\varepsilon}} * \overline{L_2^{\varepsilon}}$ and so $\overline{L_2^{\varepsilon}} - L_2^{\varepsilon} = \overline{L_1^{\varepsilon}} - L_1^{\varepsilon} \in H_2$. The fact that $\overline{L_1^{\varepsilon}} - L_1^{\varepsilon} \in H_2$ implies $\overline{L_1} - L_1 \in H_1$, follows by applying the same argument to the isomorphism ε^{-1}.

Since B satisfies (LE1)–(LE3), it is an *LE*-class. □

3 Properties of affine Grassmannians

In this section we make an initial study of properties of a Grassmannian that are properties of any of its subspace complements. More precisely, given a Grassmannian Γ_i, $i = 1, 2$, defined over a field $\mathbb{F}$ with subspace $\mathcal{S}_i$ and an isomorphism

$\varphi : \Gamma_1 - \mathscr{S}_1 \to \Gamma_2 - \mathscr{S}_2$, we ask which characteristics of Γ_1 are necessarily shared by Γ_2. A large portion of these characteristics can be read off from the residue of a point. Since the cases $k \in \{1, n\}$ have been dealt with, we will now focus on the situation where $1 < k < n$.

Let Δ denote the A_n building over $\mathbb{F}$ of which Γ is the k-shadow space. The *residue* Δ_F of a flag F of Δ is the building of all flags of Δ incident to F and having type set disjoint from $\mathrm{typ}(F)$ whose incidence relation is the one induced by Δ. Set

$$\mathscr{M}^- = \{\text{objects of type } -\}$$
$$\mathscr{M}^+ = \{\text{objects of type } +\}$$
$$\mathscr{M} = \mathscr{M}^- \cup \mathscr{M}^+.$$

For $\epsilon \in \{-, +\}$, the singular subspace $[M]_0$ with $M \in \mathscr{M}^\epsilon$ is said to be of ϵ-type. The singular subspace $[M]_{0a}$ of $\Gamma - \mathscr{S}$, whenever non-empty, is also said to be of ϵ-type.

The *coarse residual geometry* at an arbitrary point P of Γ is the point-line geometry $\mathrm{C}\Gamma_P = (\mathscr{L}_P, \mathscr{M}_P)$, where $\mathscr{L}_P$ and $\mathscr{M}_P$ are the sets of flags in $\mathscr{L}$, and $\mathscr{M}$ respectively incident to P and incidence is induced by Δ. Elements of the same type are considered to be incident only when equal; we do not consider elements of $\mathscr{M}_P^-$ and $\mathscr{M}_P^+$ to be incident, although they are in Δ. Note that for $P \notin \mathscr{S}$ and any $M \in \mathscr{M}_P$, the singular subspace $[M]_{0a}$ is non-empty.

Lemma 3.1. *Let Γ be a Grassmannian of type $A_{n,k}$ with $1 < k < n$ over a field $\mathbb{F}$ and let $\mathscr{S}$ be a, possibly empty, subspace. Then the following hold.*

(a) *The intersection of two singular subspaces from $\mathscr{M}$ of the same type consists of a single point or is empty.*

(b) *The intersection of two singular subspaces from $\mathscr{M}$ of opposite type consists of the points on a line or is empty.*

(c) *The coarse residual geometry of a point P is a grid with point set $\mathscr{L}_P$ and in which $\mathscr{M}^-$ and $\mathscr{M}^+$ form the two parallel classes of lines.*

(d) *The elements of $\mathscr{M}$ induce the only maximal singular subspaces of Γ and $\Gamma - \mathscr{S}$ alike.*

Clearly any isomorphism $\Gamma_1 \to \Gamma_2$ sends maximal singular subspaces to maximal singular subspaces and the same holds for isomorphisms $\Gamma_1 - \mathscr{S}_1 \to \Gamma_2 - \mathscr{S}_2$. Therefore using Part (d) of Lemma 3.1 the elements of $\mathscr{M}$ are well-defined objects of the point-line geometry Γ and, similarly, those elements of $\mathscr{M}$ having non-empty intersection with $\Gamma - \mathscr{S}$ are well-defined objects of the point-line geometry $\Gamma - \mathscr{S}$.

Proof of Lemma 3.1. (a) and (b): This follows from the definitions of Δ and Γ and some elementary linear algebra.

(c): This follows immediately from (a) and (b).

(AutoPDF V7 11/1/06 10:41) WDG (170×240mm) TMaths J-1445 Adv. in Geom. 6:2 PMU:I(CKN[V])10/1/2006 pp. 225–241 1445_6-2_04 (p. 231)

(d): We show that any singular subspace S of $\Gamma - \mathscr{S}$ containing at least two intersecting lines is contained in a singular subspace of type $-$ or $+$.

We first prove the following. Fix a point $P \in \Gamma - \mathscr{S}$. Since the subspace $\mathscr{S}$ intersects any line L on P in at most one point, and Γ is thick, there is at least one point Q on $L - \mathscr{S}$ different from P. Take another line M on P and let R be a point on $M - \mathscr{S}$ different from P. We claim that Q and R are collinear in $\Gamma - \mathscr{S}$ if and only if either $[L]_- = [M]_-$ or $[L]_+ = [M]_+$.

The "if" part of this claim is clearly true since the point sets $[[L]_-]_{0a}$ and $[[L]_+]_{0a}$ are singular. To prove the "only if" part, we note that a similar statement holds for Γ itself. Assume that Q and R belong to some line N. Looking at the dimensions of the intersections among $[M]_-$, $[N]_-$, $[L]_-$, $[M]_+$, $[N]_+$, and $[L]_+$, reveals that either $[M]_- = [N]_- = [L]_-$ or $[M]_+ = [N]_+ = [L]_+$.

Now Part (d) follows from our claim. Namely, let $P \in S$ be the intersection point of the two lines L and M contained in S by assumption. Let Q and R be as above. Then by our claim either $P, Q, R \in [X]_0$ where $\{X\} = [L]_-$ or $\{X\} = [L]_+$. Suppose without loss of generality that the former is true. Considering any other point $T \in S$ on some line M' with P we find that also $P, T \in [X]_0$, where $\{X\} = [M']_-$. Hence $S \subseteq [X]_0$. □

We now consider isomorphisms between affine Grassmannians.

Corollary 3.2. *For $i = 1, 2$, let Γ_i be a geometry of type A_{n_i,k_i} over a field $\mathbb{F}$ with subspace $\mathscr{S}_i$. If there is an isomorphism $\varphi : \Gamma_1 - \mathscr{S}_1 \to \Gamma_2 - \mathscr{S}_2$, then $n_1 = n_2$ and $k_1 \in \{k_2, n_2 + 1 - k_2\}$.*

Proof. This is implicit in Theorem 2 of Blok [1] for all fields $\mathbb{F}$ other than $\mathbb{F}_2$. Now let $\mathbb{F} = \mathbb{F}_2$. For $i = 1, 2$, consider a point $P_i \in \Gamma_i - \mathscr{S}_i$ and suppose $P_1^{\varphi} = P_2$. Now φ sends the coarse residual geometry of P_1 to the coarse residual geometry of P_2. It preserves the lines of the grid because φ sends maximal singular subspaces on P_1 to maximal singular subspaces on P_2. By the intersection properties between the two types of lines in the grid (Lemma 3.1 (a) and (b)), φ sends lines of the same type to lines of the same type. In the coarse residual geometry of P_i, the lines of type $+$ have size $2^{k_i} - 1$ and lines of type $-$ have size $2^{n_i+1-k_i} - 1$. Thus $\{k_1, n_1 + 1 - k_1\} = \{k_2, n_2 + 1 - k_2\}$ and $n_1 = k_1 + (n_1 + 1 - k_1) - 1 = k_2 + (n_2 + 1 - k_2) - 1 = n_2$. We are done. □

The following connectivity result is needed to show that certain local information is in fact global information.

Lemma 3.3. *For any thick Grassmannian Γ with subspace $\mathscr{S}$, the geometry $\Gamma - \mathscr{S}$ is connected.*

We note that this was proved in Shult [6] in case $\mathscr{S}$ is a hyperplane.

Proof. Let Γ be a geometry of type $A_{n,k}$ over a field $\mathbb{F}$. We prove that for any pair of distinct points P and Q in $\Gamma - \mathscr{S}$, there is an l and a path of points $P = P_0, P_1, \dots, P_l = Q$ all in $\Gamma - \mathscr{S}$ such that, for $i = 1, 2, \dots, l$ the points P_{i-1} and P_i are collinear. We use induction on n. This is clearly true for $n = 1, 2$ since in that case Γ is singular.

Now let $n \geqslant 3$. Suppose first that P and Q are incident with some m-object X of Δ. Assume without loss of generality that $m > k$. Then the subspace Γ' of Γ induced on $[X]_0$ is a geometry of type $A_{m-1,k}$. Now $\mathscr{S}' = [X]_0 \cap \mathscr{S}$ is a subspace of Γ' and so by induction there is a path from P to Q entirely contained in $\Gamma' - \mathscr{S}'$ and hence this path is entirely contained in $\Gamma - \mathscr{S}$.

Next assume that P and Q are not incident with any common object. Then $n = 2k - 1$. Note that P and Q are not collinear since that is only possible if $k = 1$, but then $n = 1 < 3$. Take lines L and M with $P \in L$, $Q \in M$ such that $[L]_-$ is not incident to a common object with $[M]_+$ and also $[M]_-$ is not incident to a common object with $[L]_+$. Since $n = 2k - 1$ there is a unique 2 object Z incident to $[L]_+$ and $[M]_+$. For each point R on L there is a unique point S on M such that R, S and Z share a common 1-object. Clearly S is the unique point on M closest to R.

Let us for the moment assume that both L and M meet $\mathscr{S}$. Let T be the unique point of L in $\mathscr{S}$ and let U be the unique point of M in $\mathscr{S}$. Since Γ is thick, one of the following must occur: $P' = P$ is closest to some point Q' on M different from U, $Q' = Q$ is closest to some point P' on L different from T, or there exists a point P' on L different from T that is closest to a point Q' on M different from U. By the previous case there is a path of points entirely in $\Gamma - \mathscr{S}$ from P' to Q' and this path extends to a path of points entirely in $\Gamma - \mathscr{S}$ from P to Q. In case L does not meet $\mathscr{S}$, then we can drop the condition that P' be different from T and the result follows even more easily. □

Lemma 3.4. *Let Γ_i be a thick geometry of type A_{n_i,k_i} with subspace $\mathscr{S}_i$. Suppose there is an isomorphism $\varphi : \Gamma_1 - \mathscr{S}_1 \to \Gamma_2 - \mathscr{S}_2$. Then two maximal singular subspaces X and Y of $\Gamma_1 - \mathscr{S}_1$ are of the same type if and only if X^φ and Y^φ are of the same type.*

Proof. By Lemma 3.1 X and Y are of type $-$ or $+$. Let P and Q be points of $\Gamma_1 - \mathscr{S}_1$ on X and Y respectively. By definition of the lines of Γ_1 there exist lines L and M in Γ_1 such that $P \in L$, $Q \in M$, and $X \in \{[[L]_-]_{0a}, [[L]_+]_{0a}\}$, $Y \in \{[[M]_-]_{0a}, [[M]_+]_{0a}\}$. By connectedness of Γ_1 (Lemma 3.3) there is an l and a path $L = L_1, P_1, L_2, \dots, P_{l-1}, L_l = M$ in $\Gamma_1 - \mathscr{S}_1$ where L_i and L_{i+1} are lines on the point P_i for all $i = 1, 2, \dots, l-1$. Now for each $i \in \{1, 2, \dots, l-1\}$ and any $\delta, \epsilon \in \{-, +\}$, we have $P_i \in [[L_i]_\epsilon]_{0a} \cap [[L_{i+1}]_\delta]_{0a}$ and so by Lemma 3.1 this intersection has size 1 if and only if $\epsilon = \delta$. In particular, both X and Y are of the same type ϵ if and only if $|X \cap [[L_2]_\epsilon]_{0a}| = 1 = |Y \cap [[L_l]_\epsilon]_{0a}|$. Under φ this all carries over, and we are done. □

Thus not only are the elements of $\mathscr{M}$ having non-empty intersection with $\Gamma - \mathscr{S}$ well-defined objects of $\Gamma - \mathscr{S}$, but their partition $\mathscr{M} = \mathscr{M}^- \uplus \mathscr{M}^+$ is a feature of $\Gamma - \mathscr{S}$ as well. Note that Lemma 3.4 does not claim that $\mathscr{M}^+$ and $\mathscr{M}^-$ separately are features of $\Gamma - \mathscr{S}$, though this is clearly the case if $n \neq 2k + 1$.

4 Line-Grassmannians

We now turn our attention to the class $\mathsf{A}_2(2)$ of Grassmannians of type $A_{n,2}$ over $\mathbb{F}_2$. We consider a geometry Γ of type $A_{n,2}$ over $\mathbb{F}_2$ with hyperplane $\mathscr{S}$. Although the situation is not as degenerate as for the class $\mathsf{A}_1(2)$, there are cases where $\mathrm{Aut}(\Gamma - \mathscr{S}) \not\leqslant \mathrm{Stab}_{\mathrm{Aut}(\Gamma)}(\Gamma - \mathscr{S})$, and we explicitly describe each case. This is done by studying the universal embedding of Γ, on which $\mathrm{Aut}(\Gamma)$ acts as a linear group, and a natural embedding of the space of maximal singular subspaces of $\Gamma - \mathscr{S}$ on which $\mathrm{Aut}(\Gamma - \mathscr{S})$ acts as a linear group.

4.1 Universal embeddings. Let $\Pi = (\mathscr{O}, \mathscr{K})$ be a partial linear space of order 2 (that is, three points per line). A *representation* of Π in the vector space V over $\mathbb{F}_2$ is a map $\phi : \mathscr{O} \to V - 0$ such that $\phi(x) + \phi(y) + \phi(z) = 0$ in V whenever $\{x, y, z\} \in \mathscr{K}$. The representation is an *embedding* if ϕ is injective, and it is *full* if $\phi(\mathscr{O})$ spans V.

Theorem 4.1. *Every partial linear space $\Pi = (\mathscr{O}, \mathscr{K})$ of order 2 has a universal full representation $\hat{}: \mathscr{O} \to \hat{V}$ over $\mathbb{F}_2$.*

If Π has an embedding, then $\hat{}$ is an embedding. In this case $\mathrm{Aut}(\Pi)$ is isomorphic to $\mathrm{Stab}_{\mathrm{PGL}(\hat{V})}(\hat{\Pi})$, the stabilizer of $\hat{\mathscr{O}}$ and $\hat{\mathscr{K}}$ in $\mathrm{PGL}(\hat{V})$.

Proof. This fundamental observation is due to Ronan [5]. Let $\hat{V}_0$ have as $\mathbb{F}_2$-basis $\hat{x}$, for $x \in \mathscr{O}$; and set $\hat{R} = \langle \hat{x} + \hat{y} + \hat{z} \mid \{x, y, z\} \in \mathscr{K} \rangle$. Then $\hat{V} = \hat{V}_0 / \hat{R}$. □

4.2 Automorphisms of affine line-Grassmannians.

Theorem 4.2. *Let Γ be a Grassmann space of type $A_{n,2}(\mathbb{F})$, where $\mathbb{F}$ is a field. Then, for each geometric hyperplane $\mathscr{S}$ of Γ, there is a symplectic form b on $V = \mathbb{F}^{n+1}$ for which $\mathscr{S}$ is the set of b-isotropic 2-spaces of V.*

Proof. This is a special case of Theorem 1 of [6] and is also due to Cooperstein and Shult [3]. □

Proposition 4.3. *Let V and b be as in Theorem 4.2 with $\mathbb{F} = \mathbb{F}_2$. Let $R = \mathrm{Rad}(V, b)$ with $\dim R = k$ and $\dim V/R = 2m (\geqslant 2)$. Let Π be the embeddable partial linear space of order 2 whose point set is $V - R$ and whose lines are the hyperbolic lines (2-spaces) for b.*

(1) *If $m = 1$, then V is a universal embedding space for Π.*

(2) *Assume $m \geqslant 2$. Then the universal embedding space $\hat{V}$ for Π has dimension $k + 2m + 1$. There is a quadratic form $\hat{q} : \hat{V} \to \mathbb{F}_2$ (with associated symplectic form $\hat{b}$) for which*

$$\hat{\mathscr{O}} = \{x \in \hat{V} \mid \hat{q}(x) = 1,\ x \notin \mathrm{Rad}(\hat{V}, \hat{b})\},$$

and $\mathscr{K}$ consists of all totally nonsingular lines (2-spaces) in $\hat{V}$ for q.

Here the singular radical $\mathrm{SRad}(\hat{V},\hat{q}) = \{x \in \mathrm{Rad}(\hat{V},\hat{b}) \mid \hat{q}(x) = 0\}$ *has dimension k and codimension* 1 *in* $\mathrm{Rad}(\hat{V},\hat{b})$, *and* $\hat{q}$ *induces a nonsingular but degenerate quadratic form on* $\hat{V}/\mathrm{SRad}(\hat{V},\hat{q})$.

Proof. See [4, Theorems 1 and 3]. □

The case $m = 1$ is that of an attenuated hyperplane of the line-Grassmannian. The hyperplane is the set of all lines of V meeting a given codimension-2 space.

Proposition 4.4. *Let* $\Gamma - \mathscr{S}$ *be the hyperplane complement of Theorem* 4.2 *with* $n \geqslant 4$, *and let* $\Pi = (\mathscr{O}, \mathscr{K})$ *be the associated partial linear space of Proposition* 4.3.
Then $\mathrm{Aut}(\Gamma - \mathscr{S}) \cong \mathrm{Aut}(\Pi)$.

Proof. The points of $\Gamma - \mathscr{S}$ are the lines of Π, and a line of $\Gamma - \mathscr{S}$ consists of two concurrent (and coplanar) lines of Π. Therefore $\Gamma - \mathscr{S}$ is the line graph of Π, and $\mathrm{Aut}(\Pi) \leqslant \mathrm{Aut}(\Gamma - \mathscr{S})$. On the other hand, by Lemma 3.4, the points of $\mathscr{O}$ can be recognized as the singular subspaces in $\Gamma - \mathscr{S}$ of maximal cardinality; so $\mathrm{Aut}(\Gamma - \mathscr{S}) \leqslant \mathrm{Aut}(\Pi)$. □

Using the notation introduced in this section, we formulate the following result.

Theorem 4.5.

(1) $\mathrm{Aut}(\Gamma - \mathscr{S}) \geqslant \mathrm{Stab}_{\mathrm{PGL}(V)}(\Gamma - \mathscr{S}) \cong 2^{2mk} : (\mathrm{GL}_k(2) \times \mathrm{Sp}_{2m}(2))$.

(2) *If* $m = 1$, *then* $\mathrm{Aut}(\Gamma - \mathscr{S}) \cong \mathrm{Aut}(\Pi) = \mathrm{Stab}_{\mathrm{PGL}(\hat{V})}(\Pi) \cong \mathrm{Stab}_{\mathrm{PGL}(V)}(\Gamma - \mathscr{S}) \cong 2^{2k} : (\mathrm{GL}_k(2) \times \mathrm{Sym}(3))$.

(3) *If* $m \geqslant 2$, *then* $\mathrm{Aut}(\Gamma - \mathscr{S}) \cong \mathrm{Aut}(\Pi) = \mathrm{Stab}_{\mathrm{PGL}(\hat{V})}(\Pi) \cong 2^{2mk+k} : (\mathrm{GL}_k(2) \times \mathrm{Sp}_{2m}(2))$. *The subgroup* $\mathrm{Stab}_{\mathrm{PGL}(V)}(\Gamma - \mathscr{S})$ *of* $\mathrm{Aut}(\Gamma - \mathscr{S})$ *is realized as the stabilizer of a nonsingular vector from* $\mathrm{Rad}(\hat{V},\hat{b}) - \mathrm{SRad}(\hat{V},\hat{q})$.

Proof. For (1), the containment is clear. We are looking for the symplectic group of the form b. Consider the subspace chain $0 \leqslant \mathrm{Rad}(V,b) \leqslant V$. The radical $\mathrm{Rad}(V,b)$ has dimension k, and $\mathrm{GL}_k(2)$ acts on it preserving the form b. Also, the full symplectic group is induced on the nondegenerate space $V/\mathrm{Rad}(V,b)$. Thus modulo the normal subgroup of all symplectic isometries that stabilize the chain $0 \leqslant \mathrm{Rad}(V,b) \leqslant V$ (that is, are trivial on both $\mathrm{Rad}(V,b)$ and $V/\mathrm{Rad}(V,b)$) we have $\mathrm{GL}_k(2) \times \mathrm{Sp}_{2m}(2)$. The radical has dimension k and codimension $2m$, so the full subgroup of $\mathrm{GL}(V) = \mathrm{PGL}(V)$ that stabilizes the chain $0 \leqslant \mathrm{Rad}(V,b) \leqslant V$ is elementary abelian of order $2^{k \times 2m}$. Any such map moves members of V only by elements of the radical of b, so all such maps are isometries, completing (1).

For (2) and (3), the first isomorphism comes from Proposition 4.4 and the equality from Theorem 4.1. Proposition 4.3.1 then completes (2).

By Proposition 4.3.2, for the final isomorphism in (3), we need the orthogonal group of the quadratic form $\hat{q}$. Its structure follows, as in (1), from consideration of the subspace chain $0 \leqslant \mathrm{SRad}(\hat{V}, \hat{q}) \leqslant \hat{V}$. Again the singular radical of dimension k admits all transformations of $\mathrm{GL}_k(2)$ as isometries. The quotient $\hat{V}/\mathrm{SRad}(\hat{V}, \hat{q})$ is a nonsingular orthogonal space of dimension $2m+1$ and all $O_{2m+1}(2) \cong \mathrm{Sp}_{2m}(2)$ acts; so the stablizer quotient is $\mathrm{GL}_k(2) \times \mathrm{Sp}_{2m}(2)$, as claimed. The full stabilizer of the chain $0 \leqslant \mathrm{SRad}(\hat{V}, \hat{q}) \leqslant \hat{V}$ is elementary abelian of order $2^{k \times (2m+1)}$. As all its elements move members of $\hat{V}$ only by members of the singular radical $\mathrm{SRad}(\hat{V}, \hat{q})$, they are all isometries of $\hat{q}$, giving (3). $\square$

Proof of Theorem 1. For $i = 1, 2$, let Γ_i be a geometry of type $A_{n_i,2}$ over $\mathbb{F}_2$ with hyperplane $\mathscr{S}_i$. If $\varepsilon : \Gamma_1 - \mathscr{S}_1 \to \Gamma_2 - \mathscr{S}_2$ is an isomorphism, then $n_1 = n_2 = n$ by Corollary 3.2.

If η is another such isomorphism, then $\eta^{-1} \circ \varepsilon \in \mathrm{Aut}(\Gamma_1 - \mathscr{S}_1)$. It now follows from Theorem 4.5 that if $n \geqslant 4$ there are certain choices of $\mathscr{S}_1$ for which there are many isomorphisms ε that are not extendable to an isomorphism $\Gamma_1 \to \Gamma_2$. $\square$

We can now answer question 1.2 of Shult [7].

Theorem 4.6. *Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be hyperplanes of the (n,k)-Grassmannian Γ with underlying vector space V, and suppose that the affine Grassmannians $\Gamma - \mathscr{S}_1$ and $\Gamma - \mathscr{S}_2$ are isomorphic. Then there is an element of $\mathrm{P\Gamma L}(V)$ that induces an isomorphism of $\Gamma - \mathscr{S}_1$ and $\Gamma - \mathscr{S}_2$.*

Proof. If V is not defined over $\mathbb{F}_2$ or if $3 \leqslant k \leqslant n-2$, then this follows immediately from Theorem 2 of [1] and the present Theorem 2.

In the remaining cases V is defined over $\mathbb{F}_2$ and we have $k = 1, 2, n-1$, or n. If $k = 1$ or n, then all hyperplanes are in the same $\mathrm{P\Gamma L}(V)$ orbit. If $k = 2$ or $n-2$, then, as seen in Theorem 4.2 above, hyperplanes correspond to symplectic forms on V. Two such forms are in the same orbit under $\mathrm{P\Gamma L}(V)$ if and only if they have the same dimension radical. But it is easy to check that if $\Gamma - \mathscr{S}_1$ and $\Gamma - \mathscr{S}_2$ are isomorphic, then the radicals of the corresponding forms have the same dimension; for instance using that the 1-spaces off the radical are the maximal singular subspaces of $\Gamma - \mathscr{S}_i$ of $-$-type. $\square$

5 Affinely rigid Grassmannians over $\mathbb{F}_2$

We now address the class of remaining Grassmannians. This is the class $\mathsf{A}_{\geqslant 3}(2)$ of (n,k)-Grassmannians over $\mathbb{F}_2$, where $3 \leqslant k \leqslant n-2$. We will label the objects of the A_n-building as follows:

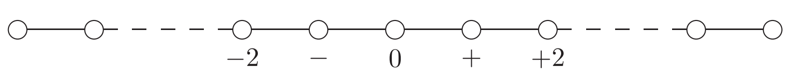

Figure 1. The central labeling of the A_n diagram

(AutoPDF V7 11/1/06 10:41) WDG (170×240mm) TMaths J-1445 Adv. in Geom. 6:2 PMU:I(CKN[V])10/1/2006 pp. 225–241 1445_6-2_04 (p. 236)

Lemma 3.1 tells us that the coarse residual geometry of a point is a grid, where $\mathscr{M}^-$ and $\mathscr{M}^+$ are the two parallel classes of lines. In order to prove Theorem 2 we will need to analyze the residue of a point in more detail. To his end define

$$\Lambda^- = \{(-,+2) \text{ flags}\}$$
$$\Lambda^+ = \{(-2,+) \text{ flags}\}$$
$$\Lambda = \Lambda^- \cup \Lambda^+$$
$$\Pi^- = \begin{cases} \{(-,+3) \text{ flags}\} & \text{if } k \leqslant n-3 \\ \{- \text{ flags}\} & \text{if } k = n-2 \end{cases}$$
$$\Pi^+ = \begin{cases} \{(-3,+) \text{ flags}\} & \text{if } 4 \leqslant k \\ \{+ \text{ flags}\} & \text{if } 3 = k \end{cases}$$
$$\Pi = \Pi^- \cup \Pi^+$$
$$\mathscr{Q} = \begin{cases} \{(-2,+2) \text{ flags}\} & \text{if } 3 \leqslant k \leqslant n-2 \\ \{-2 \text{ flags}\} & \text{if } 3 \leqslant k = n-1 \\ \{+2 \text{ flags}\} & \text{if } 2 = k \leqslant n-2 \\ \Gamma & \text{if } 2 = k = n-1 \end{cases}$$

The elements of $\mathscr{Q}$ are the symplecta of Γ. Such an element Q is a well-defined object of Γ since it is the convex closure of any two of its points at mutual distance 2. Thus given Grassmannians Γ_i with set of symplecta $\mathscr{Q}_i$, $i = 1,2$, any isomorphism $\Gamma_1 \to \Gamma_2$ necessarily maps an element of $\mathscr{Q}_1$ to an element of $\mathscr{Q}_2$. It will be important for us to decide when $Q - \mathscr{S}$, if not always, is a well-defined object of $\Gamma - \mathscr{S}$. In order to do this, we will take a closer look at the residue of a point.

For a point P, let Λ_P be the set of flags in Λ incident to P. Similarly, define Π_P, $\mathscr{M}_P$, $\mathscr{Q}_P$, Λ_P^-, Λ_P^+, Π_P^-, Π_P^-, $\mathscr{M}_P^-$, $\mathscr{M}_P^-$, and $\mathscr{Q}_P^-$.

The *residual geometry* of a point P is the point-line geometry $\Gamma_P = (\mathscr{L}_P, \Lambda_P)$ in which incidence is inherited from Δ. Again, elements of the same type are only considered incident when equal; elements from Λ_P^- and Λ_P^+ are never incident in Γ_P even if they are in Δ.

We briefly interpret the elements of Π_P, $\mathscr{M}_P$, and $\mathscr{Q}_P$ as subspaces of Γ_P and describe the relation between Γ_P and $C\Gamma_P$. To begin with, the points of Γ_P are also the points of $C\Gamma_P$. An element X from Π_P defines a subspace of Γ_P isomorphic to a projective plane denoted X_P. Similarly, an element X from $\mathscr{M}_P$, which is a line in $C\Gamma_P$, defines a subspace of Γ_P isomorphic to a projective space, also denoted X_P. An element $Q \in \mathscr{Q}$ induces a subspace of Γ of type $A_{3,2}$. Given a point P in Q, the subspace Q_P of the residual geometry Γ_P is a grid containing nine points of $\mathscr{L}_P$, and three lines from Λ_P^- and Λ_P^+ each. Since the points of Γ_P and $C\Gamma_P$ are the same and each element of Λ_P^ϵ is contained in a unique element of $\mathscr{M}_P^\epsilon$ we can regard Q_P as a small 3×3 subgrid of the larger grid $C\Gamma_P$.

For the rest of this section, let $\mathscr{S}$ be a fixed but arbitrary subspace of Γ. Assume that $P \in \mathscr{S}$. By $\mathscr{S}_P$ we denote the set of all points of Γ_P that, viewed as lines of Γ, are entirely contained in $\mathscr{S}$. This forms a subspace of Γ_P, but not necessarily of $C\Gamma_P$. Hence given some element $M \in \mathscr{M}$ containing P, the intersection $M_P \cap \mathscr{S}_P$ is a subspace of the projective subspace M_P of Γ_P. Similarly, given some element $Q \in \mathscr{Q}_P$, the intersection $Q_P \cap \mathscr{S}_P$ is a subspace of the grid-subspace Q_P of Γ_P.

The next lemma describes a convexity property of subspaces $Q - \mathscr{S}$ of $\Gamma - \mathscr{S}$ for $Q \in \mathscr{Q}$. Note that an element of $\mathscr{Q}$ is a $(3,2)$-Grassmannian. We prefer to consider it as the polar geometry $O_6^+(2)$ in its natural embedding.

We recall the notion of 2-convexity from Blok [1]. Given a point-line geometry A we call a set of points X 2-*convex* (in A) if it has the property that, for any $x, y \in X$ at mutual distance at most 2, all points on a geodesic of A from x to y are also contained in X. The 2-*convex closure* of X is the smallest 2-convex subspace containing X. The 2-convex closedness of subspace complements will be of crucial importance. Note that if A is such a subspace complement of Γ, then the 2-convex closure of a point subset of A means the closure in A, but not in Γ. Note that in that case geodesics of A need not be geodesics of Γ, but geodesics of length 2 are.

Lemma 5.1. *Let Γ be a geometry of type $O_6^+(2)$ and let $\mathscr{S}$ be a subspace. Then $\Gamma - \mathscr{S}$ is the* 2*-convex closure of any two of its points at mutual distance two*, *except if $\mathscr{S}$ is the hyperplane carrying a polar space of type $O_5(2)$.*

Proof. We will only outline the proof since almost all was done in Blok [1]. By Lemma 5.8 of Blok [1], we only have to check the case where $\mathscr{S}$ is a proper subspace of the hyperplane carrying the structure of a polar space of type $O_5(2)$. Such a subspace either is a set of pairwise non-collinear points, is contained in a hyperplane of $O_5(2)$ of type $O_4^+(2)$ (a grid), or is contained in $X^\perp$ for some point X of $O_5(2)$. The former case is dealt with in the proof of the same Lemma 5.8. In the latter two cases, $\mathscr{S}$ is in fact contained in a hyperplane of $O_6^+(2)$ of type $X^\perp$ for some point X in $O_6^+(2)$ and we are led back to Lemma 5.8. □

A *square* in a point-line geometry is a set of four points P_1, P_2, P_3, P_4 in which any two points are collinear except for the pairs (P_1, P_3) and (P_2, P_4). For the sequel it will be helpful to verify that in Γ_P every square is contained in a 3×3 grid Q_P for some $Q \in \mathscr{Q}_P$. With apologies to the reader for its distinctly ad-hoc nature, we now introduce the notion of a (*local*) *affine square*. This is a square in Γ_P (and in $C\Gamma_P$), no point of which belongs to $\mathscr{S}_P$. The only excuse for introducing it is the following useful signalling function these affine squares have.

Corollary 5.2. *Let $P \in \mathscr{S}$ and let $Q \in \mathscr{Q}_P$. If Q_P contains an affine square*, *then $Q - \mathscr{S}$ is the* 2*-convex closure of any two of its points at mutual distance* 2*.*

Proof. If $Q \cap \mathscr{S}$ is the hyperplane of type $O_5(2)$, then $Q_P \cap \mathscr{S}_P$ consists of three pairwise non-collinear points in the 3×3 grid Q_P, for any point $P \in Q$, so Q_P does not contain an affine square. The result follows now from Lemma 5.1. □

(AutoPDF V7 11/1/06 10:41) WDG (170×240mm) TMaths J-1445 Adv. in Geom. 6:2 PMU:I(CKN[V])10/1/2006 pp. 225–241 1445_6-2_04 (p. 238)

Our next aim is to prove a couple of lemmas about affine squares. They are easily interpreted by viewing the lines of the grid $C\Gamma_P$ as projective subspaces of Γ_P.

Lemma 5.3. *Fix $P \in \mathscr{S}$, non-deep and fix $\epsilon \in \{-,+\}$. If each of $M^1, M^2 \in \mathscr{M}_P^\epsilon$ contains some point of $\Gamma_P - \mathscr{S}_P$, then they contain points of $\Gamma_P - \mathscr{S}_P$ that are collinear.*

Proof. Since M_P^1 contains a point of $\Gamma_P - \mathscr{S}_P$ the set $M_P^1 \cap \mathscr{S}_P$ is a proper subspace of M_P^1 whose size is strictly less than half the size of M_P^1. The same holds for M^2. Now M^1 and M^2 are two parallel lines in the grid that is the coarse residual geometry and the result is obvious. □

Note that the next lemma is not valid when allowing $k = 2$ or $k = n - 1$. Also, in these cases it is easy to find counterexamples to the conclusions of Lemmas 5.5 and 5.6.

Lemma 5.4. *Fix $P \in \mathscr{S}$, non-deep, and let $M \in \mathscr{M}_P$. Then any pair of points from $M_P - \mathscr{S}_P$ belongs to some affine square.*

Proof. In this proof $\epsilon \in \{-,+\}$ will be a sign. Without loss of generality we may assume $\epsilon = -$. Assume $M^1 = M \in \mathscr{M}_P^\epsilon$ and let $N^1, N^2 \in \mathscr{M}_P^{-\epsilon}$ be such that the points (M^1, N^1) and (M^1, N^2) both belong to $\Gamma_P - \mathscr{S}_P$.

Fix an arbitrary $N \in \mathscr{M}_P^{-\epsilon}$. As we know, N_P is a projective space of dimension at least 2 since $3 \leqslant k$ (and in case $\epsilon = +$ since $3 \leqslant n + 1 - k$). The projection map $\pi_i : N_P^i \to N_P$, $i = 1, 2$, sending (M, N^i) to (M, N) for all $M \in \mathscr{M}_P^\epsilon$ is an isomorphism. Thus $\pi_1(N_P^1 \cap \mathscr{S}_P)$ and $\pi_2(N^2 \cap \mathscr{S}_P)$ are subspaces of N_P whose union is (contained in) the union of two hyperplanes of N_P. As N_P is a projective space of dimension at least 2, the complement of two hyperplanes contains at least two points. One of these points is (M^1, N). Let (M^2, N) with $M^2 \in \mathscr{M}_P^\epsilon$ be another such point, then both (M^2, N^1) and (M^2, N^2) belong to $\Gamma_P - \mathscr{S}_P$ and together with (M^1, N^1) and (M^1, N^2) they form the four points of an affine square. □

Lemma 5.5. *Fix $P \in \mathscr{S}$, non-deep. Then the graph with vertex set $\mathscr{L}_P - \mathscr{S}_P$ and in which two vertices are adjacent whenever the corresponding points are in an affine square, is connected.*

Proof. By Lemma 5.4 this graph is connected if (and only if) the collinearity graph of $\Gamma_P - \mathscr{S}_P$ is connected. It follows immediately from Lemma 5.3 that the latter graph is connected. In fact one easily verifies that the point-affine square graph has diameter at most 3. □

Lemma 5.6. *Fix $P \in \mathscr{S}$, non-deep. Then any point of Γ_P belongs to some grid Q_P containing an affine square.*

Proof. In this proof we denote the points of Γ_P by lower case letters. If the point under consideration is in $\Gamma_P - \mathscr{S}_P$, this follows from Lemma 5.5. Therefore, let l^0

be any point of $\mathscr{S}_P$. Since P is non-deep, there is some point l^1 in $\Gamma_P - \mathscr{S}_P$. Let $M^0 \in \mathscr{M}_P^-$ and $N^0 \in \mathscr{M}_P^+$ be such that l^0 is the line (M^0, N^0). There are two cases to distinguish.

(1) At least one of M_P^0 and N_P^0 contains a point of $\Gamma_P - \mathscr{S}_P$. Suppose M_P^0 contains a point m^0 of $\Gamma_P - \mathscr{S}_P$. Since M_P^0 is a projective space of dimension at least 2 in Γ_P with proper subspace $M_P^0 \cap \mathscr{S}_P$, there is a line of M_P^0 on l^0 and m^0 that contains another point m^1 of $\Gamma_P - \mathscr{S}_P$. By Lemma 5.4 there is an affine square containing m^0 and m^1. This affine square determines a unique grid Q_P which contains l^0.

(2) Neither M_P^0 nor N_P^0 contains a point of $\Gamma_P - \mathscr{S}_P$. Now l^0 and l^1 are contained in a unique grid Q_P. In this grid, the two lines intersecting at l^0 necessarily form the proper subspace $Q_P \cap \mathscr{S}_P$ of Q_P. Its complement is the desired affine square. □

Proof of Theorem 2. We show that $\mathsf{A}_{\geqslant 3}(2)$ satisfies the conditions of Theorem 2.5.

(LE1): By definition of a parapolar space, $\mathsf{A}_{\geqslant 3}(2)$ satisfies (LE1).

(LE2): By a result of Shult [6] (see also Blok [1, Lemma 2.1]) the class of all strong parapolar spaces and in particular $\mathsf{A}_{\geqslant 3}(2)$ satisfies (LE2).

For any $\Gamma \in \mathsf{A}_{\geqslant 3}(2)$ with subspace $\mathscr{S}$, let

$$\mathscr{T}(\mathscr{S}) = \{Q \in \mathscr{Q} \mid Q - \mathscr{S} \text{ is the 2-convex closure of any two of its points at mutual distance 2}\}.$$

We will need the following characterization of $Q - \mathscr{S}$ for $Q \in \mathscr{T}(\mathscr{S})$. By Lemma 5.1 if $Q \in \mathscr{T}(\mathscr{S})$, then $Q \cap \mathscr{S}$ can be anything other than a hyperplane of Q of type $O_5(2)$. One easily verifies that if $Q \cap \mathscr{S}$ is of type $O_5(2)$, then the 2-convex closure of any two points at mutual distance 2 in $Q - \mathscr{S}$ is a set of six points forming the vertices of an octahedron. In fact this means that the 2-convex closure in $Q - \mathscr{S}$ of any two of its points at mutual distance 2 is *not* a set of six points forming the vertices of an octahedron if and only if $Q \in \mathscr{T}(\mathscr{S})$.

We claim that $\mathscr{T}(\mathscr{S})$ satisfies conditions (L), (IL), (T), and (LE) of Theorem 2.5.

(L) and (IL) follow immediately from Lemmas 5.6 and 5.5 respectively, applied to the residual geometry of the non-deep point P.

(T): The argument will rely on the following two observations. Let $i = 1, 2$.

(1) By definition of a strong parapolar space, a symplecton T_i is convex in Γ_i, so any geodesic in Γ_i between points of T_i is contained in T_i.

(2) Moreover, any geodesic in $\Gamma_i - \mathscr{S}_i$ between points at mutual distance 2 is a geodesic in Γ_i and the same holds if we replace Γ_i by T_i. Combining this with the previous observation, we find that the 2-convex closure in $T_i - \mathscr{S}_i$ of a set of points equals the 2-convex closure of that set of points in $\Gamma_i - \mathscr{S}_i$.

Let $\Gamma_i \in \mathsf{A}_{\geqslant 3}(2)$ with subspace $\mathscr{S}_i$ $(i = 1, 2)$ and let $\varepsilon : \Gamma_1 - \mathscr{S}_1 \to \Gamma_2 - \mathscr{S}_2$ be some isomorphism. Suppose $T_1 \in \mathscr{T}(\mathscr{S}_1)$. For every $T_1 \in \mathscr{T}(\mathscr{S}_1)$, by definition $T_1 - \mathscr{S}_1$ is the 2-convex closure of any two of its points at mutual distance 2. By observation (2), this is true also if we consider the 2-convex closure in $\Gamma_1 - \mathscr{S}_1$ instead of in $T_1 - \mathscr{S}_1$. Let X, Y be points at mutual distance 2 in $T_1 - \mathscr{S}_1$. Such points exist since $T_1 - \mathscr{S}_1$ is non-degenerate by Lemma 4.15 in Blok [1]. Then, X^ε, Y^ε are points at mutual distance 2 in $\Gamma_2 - \mathscr{S}_2$. Hence, they are also at mutual distance 2 in Γ_2. Their (2-) convex hull in Γ_2 is (contained in) a symplecton T_2. By observation (1), the points X^ε and Y^ε are at mutual distance 2 in $T_2 - \mathscr{S}_2$. Therefore, since ε is an isomorphism, we find that $(T_1 - \mathscr{S}_1)^\varepsilon \subseteq T_2 - \mathscr{S}_2$. Now clearly the 2-convex closure of the points X^ε and Y^ε in $T_2 - \mathscr{S}_2$ contains $(T_1 - \mathscr{S}_1)^\varepsilon$ and so is not merely a set of 6 points forming the vertices of an octahedron. Hence our observation following the definition of $\mathscr{T}$ implies $T_2 \in \mathscr{T}(\mathscr{S}_2)$. Applying the same argument to the map ε^{-1} shows that we must have $(T_1 - \mathscr{S}_1)^\varepsilon = T_2 - \mathscr{S}_2$.

Thus (T) is satisfied.

(LE): This is true in the strongest fashion: up to isomorphism $\mathscr{T}(\mathscr{S}_1) \cup \mathscr{T}(\mathscr{S}_2)$ only contains the polar space $O_6^+(2)$. Hence, by a result from Cohen and Shult [2] (see also Theorem 4.1 of Blok [1]) this set forms an *LE*-class. □

References

[1] R. J. Blok, Extensions of isomorphisms for affine dual polar spaces and strong parapolar spaces. *Adv. Geom.* **5** (2005), 509–532.

[2] A. M. Cohen, E. E. Shult, Affine polar spaces. *Geom. Dedicata* **35** (1990), 43–76. MR1066558 (91i:51005) Zbl 0755.51004

[3] B. N. Cooperstein, E. E. Shult, Geometric hyperplanes of Lie incidence geometries. *Geom. Dedicata* **64** (1997), 17–40. MR1432532 (98c:51013) Zbl 0868.51006

[4] J. I. Hall, Linear representations of cotriangular spaces. *Linear Algebra Appl.* **49** (1983), 257–273. MR688390 (84d:51017) Zbl 0508.51009

[5] M. A. Ronan, Embeddings and hyperplanes of discrete geometries. *European J. Combin.* **8** (1987), 179–185. MR896131 (88h:51009) Zbl 0624.51007

[6] E. Shult, Geometric hyperplanes of embeddable Grassmannians. *J. Algebra* **145** (1992), 55–82. MR1144658 (93b:51004) Zbl 0751.51002

[7] E. Shult, A Survey of (c^*, c)-geometries. Unpublished manuscript, 1996.

Received 21 April, 2004; revised 21 September, 2004

R. J. Blok, Department of Mathematics and Statistics, Bowling Green State University, Bowling Green OH 43403, USA
Email: blokr@member.ams.org

J. I. Hall, Department of Mathematics, Michigan State University, USA
Email: jhall@math.msu.edu